Problem 1 (Problem 4.2-3 of Lathi & Ding) (35 points)

You are asked to design a double-sideband-suppressed carrier (DSB-SC) modulator to generate a modulated signal \( km(t) \cos(\omega_c t + \theta) \), where \( m(t) \) is a signal band-limited to \( B \) Hertz and \( k \) is a constant. The figure below shows a DSB-SC modulator available commercially. The carrier generator is not \( \cos(\omega_c t) \), but rather \( \cos^3(\omega_c t) \). That is, the cube of \( \cos(\omega_c t) \). Explain if you will still be able to use \( \cos^3(\omega_c t) \) to generate the output of \( km(t) \cos(\omega_c t + \theta) \) as required from this modulator. You are free to choose any filter you wish (that is, any bandwidth).

(a) If \( \cos^3(\omega_c t) \) does work, describe the filter you would need to generate an output of \( km(t) \cos(\omega_c t + \theta) \)?

Answer:

\[
\cos^3(\omega_c t) = \frac{3}{4} \cos(\omega_c t) + \frac{1}{4} \cos(3\omega_c t)
\]

When modulated by \( \cos^3(\omega_c t) \), the transmitted signal contains a term proportional to \( \cos(\omega_c t) \), which is the desired modulated signal with the spectrum centered at \( f_c \). The other term has a spectrum centered at \( 3f_c \). A bandpass filter centered at \( \pm f_c \) allows the passage of the desired term but suppresses the unwanted term.

(b) Determine the signal spectra at nodes labeled “b” and “c,” and indicate the frequency bands occupied by at the spectra at both nodes.

Answer: The spectra at nodes (b) and (c) are shown in the diagram below.
What is the minimum usable value of $\omega_c$ possible?

**Answer:** In order to avoid spectral folding overlap after modulation, the minimum usable value of $f_c$ is bandwidth $B$, where $B$ is the bandwidth of the low-pass signal $m(t)$.

Would this modulator scheme work if the carrier output were changed to $\sin^3(\omega_c t)$ instead? Explain clearly the reason for your answer.

**Answer:** Yes. Since $\sin^3(\omega_c t)$ is essentially $\cos^3(\omega_c t)$ delayed by $\pi/2$, if we modulate with $\sin^3(\omega_c t) = \cos^3(\omega_c t - \pi/2)$, then we can demodulate with $\cos^3(\omega_c t - \pi/2) = \sin^3(\omega_c t)$.

Will the same modulator scheme work if the carrier generator output were $\cos^n(\omega_c t)$, for any integer $n \geq 2$, instead? If yes, why does it?

**Answer:** The expansion of $\cos^n(\omega_c t)$ contains a term like $a_1 \cos(\omega_c t)$ when $n$ is odd, but not when $n$ is even. Hence, the system works for carriers $\cos^n(\omega_c t)$ if and only if $n$ is odd.

**Problem 2 Audio Scrambler (Problem 4.2-8 Lathi & Ding)** (20 points)

The system shown below is used for scrambling audio signals (although it is not a very sophisticated scrambler). The output $y(t)$ is the scrambled version of the input $m(t)$. 
(a) Sketch the spectrum of the scrambled output signal \( y(t) \) on the graph below.

**Answer:** The spectrum \( Y(f) \) of the scrambled signal is shown in the figure below:

(b) Suggest a method for descrambling \( y(t) \) to recover \( m(t) \).

**Answer:** We observe that \( Y(f) \) is the same as \( M(f) \) with frequency spectrum inverted, that is, the high frequencies are shifted to lower frequencies and vice versa. To recover the original spectrum, we must invert \( Y(f) \), which can be done using exactly the same scrambler circuit.

**Problem 3 High-Q Resonant RLC Circuit** (25 points)

One possible frequency selective circuit is an \( LRC \) resonator as schematically shown below. The resonant frequency is given by

\[
f_{\text{resonance}} = \frac{1}{\sqrt{LC}}
\]

The Quality factor of a resonant circuit (\(Q\)-factor for short) is defined as the resonant frequency divided by the half-power (or -3 dB) bandwidth; in symbols it is

\[
Q = \frac{\text{resonance frequency}}{\text{half-power bandwidth}} = \frac{f_{\text{resonance}}}{B}
\]

In this problem our focus is on passing an amplitude modulated signal through the \( RLC \) filter and how it can affect the modulation of the carrier.
(a) Derive the transfer function $H(2\pi f)$ for this parallel RLC circuit. Assume the sinusoidal steady-state in deriving the transfer function.

Answer:

$$H(\omega) = \frac{i_R}{i(t)} = \frac{j\omega L}{R + j\omega L + (j\omega)^2 RCL}$$

$$|H(\omega)| = \left| \frac{i_R}{i(t)} \right| = \frac{\omega L}{\sqrt{(R - \omega^2 LCR)^2 + (\omega L)^2}}$$

(b) Derive the half-power bandwidth $B$ (that is, the -3 dB bandwidth) for this circuit. Express $B$ as a function of $R$, $L$ and $C$.

Answer:

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + (LC)^2} \quad \text{and} \quad \omega_2 = +\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + (LC)^2}$$

Bandwidth $B = \omega_2 - \omega_1 = \frac{1}{RC}$

(c) Starting with the definition of $Q$-factor at the beginning of the problem statement, derive an expression for $Q$ as a function of the circuit parameters.

Answer:

$$Q = \frac{\omega_0}{B} = \frac{\omega_0 RC}{\omega_0 L} = \frac{R}{\omega_0 L} \quad \text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

(d) A carrier wave of frequency of 1 MHz is modulated at 50% by a sinusoidal frequency of 5 kHz. [Note: A modulation index of 50% means the modulation signal amplitude range is 50% of the amplitude range of the carrier signal.] We assume that $L$ and $C$ have been chosen so that the circuit’s resonance frequency is exactly 1 MHz. Resistor
$R$ is chosen so the **Q-factor is 200**. If a modulated signal with 50% modulation is passed through this RLC filter network, with the carrier frequency passed with no loss, how much is the 5 kHz modulation affected by the RLC circuit? Another way to ask this question, by how much is the modulation reduced at 5 kHz by the RLC circuit?

**Answer:** We know that $\omega_0$ is equal to $2\pi f_0 = 2\pi \times 10^6$ Hz and that $Q = f_0/B = 200$. Therefore, we calculate that $B = 5,000$ Hz and the bandwidth is defined as the -3 dB power point on the transfer function’s response. In magnitude that 0.707 or one over the square root of 2. The tone modulation frequency is at 5 kHz so the band pass filter (BPF) reduces the modulation on the signal passing through it by one over the square root of 2 (equivalent to multiplying by 0.707).

Thus we find that a 50% modulation on the signal is reduced to 35.4% modulation by the skirts of the BPF.

**Problem 4 Free Space Propagation Loss** (20 points)

Wireless communications depends upon the transmission of radio waves from transmitting antenna to receiving antenna. In ES 442 we do not have the time to discuss antenna operation in much detail, but this problem illustrates an important feature of wireless communication. Point-to-point wireless (such as line-of-sight microwave links) depends upon the free space propagation model. The free space power $P_r(d)$ received by the antenna separated by distance $d$ from the transmitting antenna is given by the Friis free space equation.

$$P_r(d) = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2 d^2 L},$$

where $P_t$ is the transmitted power, $G_t$ is the transmitter antenna gain, $G_r$ is the receiver antenna gain, $d$ is the distance between antennas in meters, $L$ is the system loss not related to propagation losses ($L \geq 1$) and $\lambda$ is the carrier wavelength in meters. The two antenna gain terms depend upon the effective apertures of the antennas – for this problem we will ignore that and assume $G_t$ and $G_r$ both equal unity. The wavelength depends upon frequency and the speed of electromagnetic waves ($c = 300,000,000$ meters per second); given by the equation

$$\lambda = \frac{c}{f} = \frac{2\pi c}{\omega_c},$$

where $f$ is the frequency in Hertz, and $\omega_c$ is the carrier frequency in radians per second. The values for $P_r$ and $P_t$ must be expressed in appropriate units.

**(a)** All of this is background. If a transmitter produces 50 watts of radiated power at the antenna [$P_t (\text{dBW}) = 17 \text{ dBW} = 47 \text{ dBm}$], what is the received power in dBm at the receiver if the carrier frequency is 900 MHz (that is GSM cellular frequency) at $d = 100$ meters? Assume both antenna gains to be unity and let $L = 1$. 

\textbf{Answer:} We know that the wavelength $\lambda$ is the speed of light (300,000,000 meters/second) divided by the frequency ($f_c = 900$ MHz = $9 \times 10^8$ Hz).

$$\lambda = \frac{300,000,000}{900,000,000} = \frac{1}{3} \text{ meter}$$

$$P_r(d = 100 \text{ m}) = \frac{P_G G_r \lambda^2}{(4\pi)^2 d^2 L} = \frac{(50,000 \text{ mW})(1)(1/3 \text{ m})^2}{(4\pi)^2 (100 \text{ m})^2 (1)} = 3.5 \times 10^{-3} \text{ mW}$$

$$P_r(d = 100 \text{ m}) = -24.5 \text{ dBm}$$

\textbf{(b)} Now let the distance be 10 kilometers; what is the received power in dBm?

\textbf{Answer:}

$$P_r(d = 10 \text{ km}) = P_r(d = 100 \text{ m}) + 20\log_{10} \left( \frac{100}{10,000} \right) = -24.5 \text{ dBm} - 40 \text{ dBm}$$

\textbf{(c)} What conclusion do you draw as distance $d$ increases? How many dB per decade as distance change?

\textbf{Answer:} Since the received power $P_r(d)$ varies inversely with the square of the distance $d$, the received power decreases by 20 dB for a factor of ten increase in distance.