Review: Fourier Trigonometric Series (for Periodic Waveforms)

Periodic function $g(t)$ is defined in time interval of $t_1 \leq t \leq t_1 + T_0$

$$g(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos(n2\pi f_0 t) + b_n \sin(n2\pi f_0 t) \right]$$

where $g(t)$ is of period $T_0$, and $f_0 = \frac{1}{T_0}$.

We can calculate the coefficients from the equations:

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \cdot \cos(n2\pi f_0 t) dt \quad \text{for} \quad n = 1, 2, 3, 4, \ldots$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} g(t) \cdot \sin(n2\pi f_0 t) dt \quad \text{for} \quad n = 1, 2, 3, 4, \ldots$$
**Review: Exponential Fourier Series (for Periodic Functions)**

Again, periodic function $g(t)$ is defined in time interval $t_1 \leq t \leq t_1 + T_0$

$$g(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t} \quad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

where again $g(t)$ is of period $T_0$, and $f_0 = \frac{1}{T_0}$.

We can calculate the coefficients $\{D_n\}$ from the equation:

$$D_n = \frac{1}{T_0} \int_{T_0}^{T_0} g(t) \cdot e^{jn2\pi f_0 t} \, dt \quad \text{for} \ n \ \text{ranging from} \ -\infty \ \text{to} \ \infty.$$

But, periodicity is too limited for our needs, so we will need to proceed on to the Fourier Transform.

The reason is that signals used in communication systems are not periodic (it is said they are *aperiodic*).
Review: Time Domain and Frequency Domain

The Fourier series expresses any time-periodic function in terms of sinusoidal functions (only need to specify the frequency, amplitude and phase of each sinusoid). When you think about that it is remarkable that any periodic function can be simply represented by a sum of sinusoidal functions.

In other words, the Fourier series of a time-periodic function generates a frequency spectrum of harmonically-related sine and cosine waveforms.

"Every signal has a spectrum and is also determined by its spectrum. You can analyze the signal either in the time domain or in the frequency domain."

-- Professor Brad Osgood (Stanford University)

This leads to time duration versus frequency bandwidth duality in signals.
Sinusoidal Waveforms are the Building Blocks in the Fourier Series

Harmonic Motion Produces Sinusoidal Waveforms

Sheet of paper unrolls in this direction

Mechanical Oscillation

Time t

Past

Future

Electrical LC Circuit Oscillation

LC Tank Circuit

Amplifier (A)

180° Phase shift

Output

Electromagnetic coupling
Visualizing Time Domain & Frequency Domain of a Signal

Source: Agilent Technologies Application Note 150, Spectrum Analyzer Basics
Example: Periodic Square Wave

Odd function

\[ f(t) = \frac{4}{\pi} \left[ \sin(\pi t) + \frac{1}{3} \sin(3\pi t) + \frac{1}{5} \sin(5\pi t) + \frac{1}{7} \sin(7\pi t) + \cdots \right] \]

- Fundamental only
- Five terms
- Eleven terms
- Forty-nine terms

Example: Periodic Square Wave (continued)
Square Wave From Fundamental + 3\textsuperscript{rd} + 5\textsuperscript{th} & 7\textsuperscript{th} Harmonics

\[ f(t) = \frac{4}{\pi} \left[ \sin(\pi t) + \frac{1}{3} \sin(3\pi t) + \frac{1}{5} \sin(5\pi t) + \frac{1}{7} \sin(7\pi t) + \cdots \right] \]

\[ \theta = \pi t \]

https://en.wikipedia.org/wiki/Fourier_series
Another Example: Both Sine & Cosine Functions Required

Note phase shift in the fundamental frequency sine waveform.

http://www.peterstone.name/Maplepgs/fourier.html#anchor2315207
Gibb’s Phenomena at Discontinuities in Waveforms

The Gibbs phenomenon is an overshoot (or "ringing") of Fourier series occurring at simple discontinuities.

About 9% overshoot at discontinuities of waveform. This is an artifact of the Fourier series representation.
Fourier Series versus Fourier Transform

Fourier series for continuous-time periodic signals $\rightarrow$ discrete spectra
Fourier transform for continuous aperiodic signals $\rightarrow$ continuous spectra
Definition of Fourier Transform

\[ g(t) \leftrightarrow G(f) \quad \text{and} \quad G(t) \leftrightarrow g(-f) \]

Time-frequency duality: \( g(t) \leftrightarrow G(f) \quad \text{and} \quad G(t) \leftrightarrow g(-f) \)

We say “near symmetry” because the signs in the exponentials are different between the Fourier transform and the inverse Fourier transform.
Fourier Transform Produces a Continuous Spectrum

$FT\left[g(t)\right]$ gives a spectra consisting of a continuous sum of exponentials with frequencies from $-\infty$ to $+\infty$.

$$G(f) = |G(f)| \exp(j\theta(f))$$

where $|G(f)|$ is the continuous amplitude spectrum of $g(t)$ and $\theta(f)$ is the continuous phase spectrum of $g(t)$.

Supplemental:
Lathi & Ding pp. 97-99
Example: Rectangular Pulse

\[ g(t) = \text{rect}(t) = \Pi(t / \tau) = \begin{cases} 1 & \text{for } -\frac{\tau}{2} \leq t \leq \frac{\tau}{2} \\ 0 & \text{for all } |t| > \frac{\tau}{2} \end{cases} \]

Remember \( \omega = 2\pi f \)

Lathi & Ding pp. 101-102
Example 3.2
Two Definitions of the Sinc Function

\[
sinc(x) = \frac{\sin(x)}{x} \quad (1)
\]

and

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x} \quad (2)
\]

Sometimes \(sinc(x)\) is written as \(Sa(x)\).

Lathi and Ding, page 100, use the definition \(sinc(x) = \frac{\sin(x)}{x}\).
Definition of the Sinc Function

Unfortunately, there are two definitions of the sinc function in use.

Format 1 (used in Lathi and Ding, fourth edition – pp. 100 – 102)

\[
sinc(x) = \frac{\sin(x)}{x}
\]

Format 2 (used in many other text books)

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x}
\]

Properties:

1. \(\text{sinc}(x)\) is an even function of \(x\)
2. \(\text{sinc}(x) = 0\) at points where \(\sin(x) = 0\), that is, \(\text{sinc}(x) = 0\) when \(x = \pm \pi, \pm 2\pi, \pm 3\pi, \ldots\)
3. Using L’Hôpital’s rule, it can be shown that \(\text{sinc}(0) = 1\)
4. \(\text{sinc}(x)\) oscillates as \(\text{sinc}(x)\) oscillates and monotonically decreases as \(1/x\) decreases with increasing \(|x|\)
5. \(\text{sinc}(x)\) is the Fourier transform of a single rectangular pulse
Sinc Function in Two-Dimensions (e.g., Optics)

\[ \Delta \theta = \frac{1.22 \lambda}{D} \]

Pardon the digression!
Sinc Function Tradeoff: Pulse Duration *versus* Bandwidth

\[ g_1(t) \]

\[ G_1(f) \]

\[ g_2(t) \]

\[ G_2(f) \]

\[ g_3(t) \]

\[ G_3(f) \]

\[ T_1 > T_2 > T_3 \]

Lathi & Ding pp. 110-111
Sinc Function Appears in Both Pulse Train & a Single Pulse

Pulse Train – **Fourier Series**

Single pulse – **Fourier Transform**

What does this imply for a digital binary communication signal?
Some Insight into the Sinc Function

\[ \text{sinc}(x) = \frac{\sin(x)}{x} \]

Lathi & Ding
pp. 100-101
Properties of Fourier Transforms

1. Linearity (Superposition) Property
2. Time-Frequency Duality Property
3. Transform Duality Property
4. Time-Scaling Property
5. Time-Shifting Property
6. Frequency-Shifting Property
7. Time Differentiation & Time Integration Property
8. Area Under g(t) Property
9. Area Under G(f) Property
10. Conjugate Functions Property
Linearity (Superposition) Property

Given $g_1(t) \leftrightarrow G_1(f)$ and $g_2(t) \leftrightarrow G_2(f)$; 

Then $g_1(t) + g_2(t) \leftrightarrow G_1(f) + G_2(f)$ \hspace{1cm} (additivity) 

also $kg_1(t) \leftrightarrow kG_1(f)$ and $kg_2(t) \leftrightarrow kG_2(f)$ \hspace{1cm} (homogeneity) 

Combining these we have, 

$$kg_1(t) + mg_2(t) \leftrightarrow kG_1(f) + mG_2(f)$$ 

Hence, the Fourier Transform is a linear transformation. 

This is the same definition for linearity as used in your circuits and systems course.
Time-Frequency Duality Property

Given \( g(t) \Leftrightarrow G(f) \), then
\[ g(t - t_0) \Leftrightarrow G(f) e^{-j2\pi ft_0} \]
and
\[ g(t)e^{j2\pi ft_0} \Leftrightarrow G(f - f_0) \]

This leads directly to the transform Duality Property

Lathi & Ding
pp. 106-109
Transform Duality Property

Given \( g(t) \Leftrightarrow G(f) \), then
\[
\begin{align*}
g(t) & \Leftrightarrow G(f) \\
\text{and} & \\
G(t) & \Leftrightarrow g(-f)
\end{align*}
\]

See illustration on next page for example!
Illustration of Fourier Transform Duality

\[ g_1(t) \]
\[ \frac{-T_1}{2} \quad \frac{T_1}{2} \]
\[ f_1 = \frac{1}{T_1} \]

\[ g_2(t) \]
\[ \frac{-1}{T_1} \quad \frac{1}{T_1} \]
\[ \frac{-2}{T_1} \quad \frac{2}{T_1} \]

Lathi & Ding
Page 109
**Time-Scaling Property**

Given \( g(t) \Leftrightarrow G(f) \), then for a real constant \( a \),

\[
g(at) \Leftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right)
\]

Time compression of a signal results in spectral expansion and
time expansion of a signal results in spectral compression.

Lathi & Ding pp. 110-112
Time-Shifting Property

Given \( g(t) \Leftrightarrow G(f) \),
then \( g(t - t_0) \Leftrightarrow G(f) e^{-j2\pi ft_0} \)

Delaying a signal by \( t_0 \) seconds does not change its amplitude spectrum, but the phase spectrum is changed by \(-2\pi ft_0\). Note that the phase spectrum shift changes linearly with frequency \( f \).
Frequency-Shifting Property

aka Modulation Property

Given \( g(t) \Leftrightarrow G(f) \),
then \( g(t)e^{j2\pi ft_0} \Leftrightarrow G(f - f_0) \)

Multiplication of a signal \( g(t) \) by the factor \( \cos(2\pi f_c) \)
places \( G(f) \) centered at \( f = \pm f_c \).
Frequency-Shifting Property (continued)

Multiplication of a signal $g(t)$ by the factor $\cos(2\pi f_0)$ places $G(f)$ centered at $f = \pm f_c$.

Lathi & Ding
Figure 3.21
Page 115

Note phase shift

$G(f)$
$|G(f)|$
$\theta_g(f)$

$g(t)$
$g(t)\cos(2\pi f_0)$
$g(t)\sin(2\pi f_0)$
Time Differentiation & Time Integration Property

Given \( g(t) \Leftrightarrow G(f) \)

For time differentiation:

\[
\frac{dg(t)}{dt} \Leftrightarrow j2\pi f G(f)
\]

and \( \frac{d^n g(t)}{dt^n} \Leftrightarrow (j2\pi f)^n G(f) \)

For time integration:

\[
\int_{-\infty}^{t} g(\tau)d\tau \Leftrightarrow \frac{G(f)}{j2\pi f} + \frac{1}{2} G(0) \delta(f)
\]

Lathi & Ding pp. 121-123
Area Under $g(t)$ & Area Under $G(f)$ Properties

Given $g(t) \Leftrightarrow G(f)$,

Then $\int_{-\infty}^{\infty} g(t) \, dt = G(0)$

That is, the area under a function $g(t)$ is equal to the value of its Fourier transform $G(f)$ at $f = 0$.

Given $g(t) \Leftrightarrow G(f)$,

Then $g(0) = \int_{-\infty}^{\infty} G(f) \, df$

That is, the value of a function $g(t)$ at $t = 0$ is equal to the area under its Fourier transform $G(f)$.
Conjugate Functions Property

Given \( g(t) \Leftrightarrow G(f) \),
Then for a complex-valued time function \( g(t) \),
we have
\[
g^*(t) \Leftrightarrow G^*(-f)
\]
where the star symbol (*) denotes the complex conjugate operation.
The corollary to this property is
\[
g^*(-t) \Leftrightarrow G^*(f)
\]
End of Fourier Transform Property Slides
Some Important Fourier Transform Pairs

Impulse Function

Impulse train

Unit Rectangle Pulse

Unit Triangle Pulse

Exponential Pulse

Signum Function
The Impulse Function $\delta(t)$

$\delta(t) = 0 \quad \text{for all } t \neq 0$
Fourier Transform of the Impulse Function $\delta(t)$

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} \, dt = e^{-j2\pi f(0)} = 1$$

because the exponential function to zero power is unity. Therefore, we can write $\delta(t) \Leftrightarrow 1$

$\delta(t)$ is often called the Dirac delta function.
Inverse Fourier Transform of the Impulse Function $\delta(t)$

$$F^{-1}[\delta(f)] = \int_{-\infty}^{\infty} \delta(f)e^{j2\pi ft} df = e^{-j(0)t} = 1$$

because the exponential function to zero power is unity. Therefore, we can write \(1 \Leftrightarrow \delta(t)\)

This is just a DC signal.

A DC signal is zero frequency.
Fourier Transform of Impulse Train \( \delta(t) \)

*aka* “Dirac Comb Function,” Shah Function & “Sampling Function”

Shah function \( \Pi(t) \):

\[
\Pi(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_0) = \sum_{n=-\infty}^{\infty} \delta(t+nT_0)
\]

\[
\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \Pi(t) dt = 1
\]

\( \Pi(t) \) at \( t \):

\[ \cdots -2T_0 -T_0 0 T_0 2T_0 \cdots \]

Period = \( T_0 \)

\( f \):

\[ \cdots -2f_0 -f_0 0 f_0 2f_0 \cdots \]

Period = \( \frac{1}{T_0} \)
Shah Function (Impulse Train) Applications

The sampling property is given by

\[ \text{III}(t) f(t) = \sum_{n=\infty}^{\infty} f(n) \delta(t - nT_0) \]

The "replicating property" is given by the convolution operation:

\[ \text{III}(t) \star f(t) = \sum_{n=\infty}^{\infty} f(t - nT_0) \]

Convolution theorem:

\[ g_1(t) \star g_2(t) \iff G_1(f)G_2(f) \quad \text{and} \]

\[ g_1(t)g_2(t) \iff G_1(f) \star G_2(f) \]
Sampling Function in Operation

\[ \mathbb{I}(t) f(t) = \sum_{n=-\infty}^{\infty} f(n) \delta(t - nT_0) \]
Fourier Transform of Complex Exponentials

\[ F^{-1} \left[ \delta(f - f_c) \right] = \int_{-\infty}^{\infty} \delta(f - f_c) e^{-j2\pi f t} df \]

Evaluate for \( f = f_c \)

\[ F^{-1} \left[ \delta(f - f_c) \right] = \int_{f = f_c} e^{-j2\pi f_c t} df = e^{-j2\pi f_c t} \]

\[ \therefore \delta(f - f_c) \Leftrightarrow e^{-j2\pi f_c t} \] and

\[ F^{-1} \left[ \delta(f + f_c) \right] = \int_{-\infty}^{\infty} \delta(f + f_c) e^{-j2\pi f t} df \]

Evaluate for \( f = -f_c \)

\[ F^{-1} \left[ \delta(f + f_c) \right] = \int_{f = -f_c} e^{j2\pi f_c t} df = e^{j2\pi f_c t} \]

\[ \therefore \delta(f + f_c) \Leftrightarrow e^{j2\pi f_c t} \]
Fourier Transform of Sinusoidal Functions

Taking \( \delta(f - f_c) \Leftrightarrow e^{-j2\pi f_c t} \) and \( \delta(f + f_c) \Leftrightarrow e^{j2\pi f_c t} \)
We use these results to find FT of \( \cos(2\pi ft) \) and \( \sin(2\pi ft) \)
Using the identities for \( \cos(2\pi ft) \) and \( \sin(2\pi ft) \),

\[
\cos(2\pi ft) = \frac{1}{2} \left[ e^{j2\pi f_c t} + e^{-j2\pi f_c t} \right] \quad \& \quad \cos(2\pi ft) = \frac{1}{2j} \left[ e^{j2\pi f_c t} - e^{-j2\pi f_c t} \right]
\]

Therefore,

\[
\cos(2\pi ft) \Leftrightarrow \frac{1}{2} \left[ \delta(f + f_c) + \delta(f - f_c) \right], \quad \text{and} \\
\sin(2\pi ft) \Leftrightarrow \frac{1}{2j} \left[ \delta(f + f_c) - \delta(f - f_c) \right]
\]

\[\text{Lathi & Ding pp. 105}\]
Fourier Transform of Signum (Sign) Function

\[
\text{sgn}(t) = \begin{cases} 
+1, & \text{for } t > 0 \\
0, & \text{for } t = 0 \\
-1, & \text{for } t < 0 
\end{cases}
\]

We approximate the signum function using

\[
g(t) = \begin{cases} 
\exp(-at), & \text{for } t > 0 \\
0, & \text{for } t = 0 \\
-\exp(at), & \text{for } t < 0 
\end{cases}
\]

\[
G(f) = \frac{-j4\pi f}{a^2 + (2\pi f)^2} = \frac{1}{j\pi f}, \quad \text{because}
\]

\[
\lim_{a \to 0} \left[ \frac{-j4\pi f}{a^2 + (2\pi f)^2} \right] = \frac{-j}{\pi f} = \frac{1}{j\pi f}
\]

Lathi & Ding pp. 105-106
# Summary of Several Fourier Transform Pairs

<table>
<thead>
<tr>
<th>Time Function</th>
<th>Frequency Function</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Boxcar</strong></td>
<td>$G(t) = \begin{cases} 1, &amp;</td>
</tr>
<tr>
<td></td>
<td>$S(f) = \tau \text{sinc} (\tau f)$</td>
</tr>
<tr>
<td><strong>Triangle</strong></td>
<td>$G(t) = \begin{cases} 1 -</td>
</tr>
<tr>
<td></td>
<td>$S(f) = \frac{1}{\tau \pi} \sin (\pi f)$</td>
</tr>
<tr>
<td><strong>Gaussian</strong></td>
<td>$G(t) = e^{-\frac{t^2}{2\tau^2}}$</td>
</tr>
<tr>
<td></td>
<td>$S(f) = (\tau \pi)^{1/2} e^{-\pi^2 f^2/\tau^2}$</td>
</tr>
<tr>
<td><strong>Impulse</strong></td>
<td>$G(t) = \delta(t)$</td>
</tr>
<tr>
<td></td>
<td>$S(f) = 1$</td>
</tr>
<tr>
<td><strong>Sinusoid</strong></td>
<td>$G(t) = \cos \omega_0 t$</td>
</tr>
<tr>
<td></td>
<td>$S(f) = \frac{1}{2} \left( \delta(f + \Delta f) + \delta(f - \Delta f) \right)$</td>
</tr>
<tr>
<td><strong>Comb.</strong></td>
<td>$G(t) = \text{comb} (f)$</td>
</tr>
<tr>
<td></td>
<td>$S(f) = \sum_{n=-\infty}^{\infty} \delta(f - n/\tau)$</td>
</tr>
</tbody>
</table>

---

Lathi & Ding
Table 3.1
Page 107;
For a more complete Table of Fourier Transforms
---------
See also the Fourier Transform Pair Handout
A **spectrum analyzer** measures the magnitude of an input signal versus frequency within the full frequency range of the instrument. It measures frequency, power, harmonics, distortion, noise, spurious signals and bandwidth.

- It is an electronic receiver
- Measure magnitude of signals
- Does not measure phase of signals
- Complements time domain
Fourier Transform of Cosine Signal

\[ A \cos(2\pi f_c t) = \frac{A}{2} [\delta(f + f_c) + \delta(f - f_c)] \]

Blue arrows indicate positive phase direction.

3D View

Not in Lathi & Ding
Fourier Transform of Sine Signal

\[ B \sin(2\pi f_c t) = j \frac{B}{2} \left[ \delta(f + f_c) - \delta(f - f_c) \right] \]

We must subtract 90° from \( \cos(\cdots) \) to get \( \sin(\cdots) \)
Fourier Transform of Sine Signal (as usually shown in books)

\[ \sin(t) \]

\[ \delta(-f_0) \]

\[ j \frac{A}{2} \]

\[ -j \frac{A}{2} \]

Imaginary axis
Fourier Transform of Phase Shifted Sinusoidal Signal

\[ R \cdot e^{j\phi} e^{j2\pi ft} + R \cdot e^{-j\phi} e^{-j2\pi ft} \]

\[ R = \sqrt{\left(\frac{A}{2}\right)^2 + \left(\frac{B}{2}\right)^2} \quad \text{and} \quad \phi = \tan^{-1}\left(-\frac{A}{B}\right) \]
Even, Odd Relationships for Fourier Transform (1)

Even, Odd Relationships for Fourier Transform (2)

Effect of the Position of a Pulse on Fourier Transform

\[ |G(f)| = \sqrt{\left[ \text{Re}(G(f)) \right]^2 + \left[ \text{Im}(G(f)) \right]^2} \]

This time shifted pulse is both even and odd.

Both must be identical.
Selected References


