Costas Receiver
What it does and how it works

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The **Costas loop** is a synchronous receiver system that can be used for demodulating DSB-SC signals. This loop, and its variations, is widely-used as a method of carrier acquisition (or carrier recovery), and simultaneous message demodulation in communication systems, both analog and digital.

It consists of two coherent detectors supplied with the same input signal and a **phase lock loop** to provide a coherent internal signal.
Preliminary Note: Coherent Demodulation Output

Costas Loop Output: \( v_0(t) \sim m(t) \cdot \cos(\phi) \)

To optimize \( v_0(t) \) we need \( \phi \approx 0 \), and for \( v_0(t) \) to be proportional to \( m(t) \); hence \( \phi \approx \) constant

This requires us to synchronize the local oscillator (LO) to the carrier frequency and phase in the receiver.
Costas Receiver Block Diagram

\[ A_C m(t) \cdot \cos(2\pi f_c t) \]

\[ \cos(2\pi f_c t + \phi) \]

\[ \sin(2\pi f_c t + \phi) \]
**Costas Receiver: Coherent Demodulation**

Goals:
2. Adjust the local oscillator phase such that \( \phi = 0 \).
Goals:
(1) Coherent demodulation of the DSB-SC input signal.
(2) Adjust the local oscillator phase so that $\phi = 0$. 
Digression: What is a Phase Lock Loop? (1)

IN WORDS:
A phase-locked loop (PLL) is an electronic circuit with a voltage-controlled oscillator that is constantly being adjusted to match the frequency and phase (*i.e.*, lock on) of an input signal.

PLL APPLICATIONS:
(1) Synchronize a communications channel (locking it to a carrier, symbol rate or clock frequency)
(2) PLL can be used to generate a signal
(3) Modulate or demodulate a signal
(4) Reconstitute a signal with less noise
(5) Multiply or divide a frequency

PLLs are often used in wireless communication, especially where signals are modulated using frequency (FM), phase (PM) or amplitude (AM).
Digression: What is a Phase Lock Loop? (2)

A PLL consists of three basic components:

- Phase detector
- Loop filter
- Voltage-controlled oscillator (VCO)

PLL Diagram:

\[
\begin{align*}
A \left[ \cos \omega_c t + \theta_i(t) \right] & \hspace{2cm} H(s) \\
\text{Input freq} & \hspace{2cm} \text{Low pass filter} \\
\text{Phase detector} & \hspace{2cm} \text{Bias generator} \\
\text{Output freq} & \hspace{2cm} \text{VCO} \\
2B \left[ \cos \omega_c t + \theta_o(t) \right] & \hspace{2cm} e_o(t)
\end{align*}
\]
Costas Receiver: Phase Lock Circuit

For $\phi > 0$, local oscillator (LO) frequency needs to temporarily decrease.
Costas Receiver: Phase Lock Circuit

For $\phi < 0$, the local oscillator (LO) frequency needs to temporarily increase to catch up. Local oscillator phase lags behind carrier and its frequency must increase to catch up.
Costas Receiver: In-Phase and Quadrature-Phase

I-phase (in-phase coherent detector)

Product Modulator

\( v_I(t) \)

Low-Pass Filter

\( \cos(2\pi f_c t + \phi) \)

Voltage-controlled oscillator

Phase Discriminator

DSB-SC signal input

\( A_c m(t) \cdot \cos(2\pi f_c t) \)

Q-phase (quadrature-phase coherent detector)

Product Modulator

\( v_Q(t) \)

Low-Pass Filter

\( \sin(2\pi f_c t + \phi) \)
Costas Receiver: In-Phase Coherent Detector

\[ A_C \cos(2\pi f_C t) \cdot \cos(2\pi f_C t + \phi) \cdot m(t) \]

\[ v_I(t) = \frac{A_C}{2} \cos[2\pi(2f_C) t + \phi] \cdot m(t) + \frac{A_C}{2} \cos(\phi) \cdot m(t) \]

From \( \cos(A)\cos(B) = \frac{1}{2} \cos(A + B) + \frac{1}{2} \cos(B - A) \)
Costas Receiver: In-Phase Coherent Detector

Letting \( S(t) = A_c m(t) \cdot \cos(2\pi f_c t) \)
\[ v_1(t) = S(t) \cdot \cos(2\pi f_c t + \phi) \]

Remember
\[ \cos(2\pi f_c t + \phi) = \frac{e^{j\phi}}{2} e^{j2\pi f_c t} + \frac{e^{-j\phi}}{2} e^{-j2\pi f_c t} \]

\[ \text{FT:} \quad \frac{e^{j\phi}}{2} e^{j2\pi f_c t} + \frac{e^{-j\phi}}{2} e^{-j2\pi f_c t} \Leftrightarrow \frac{e^{j\phi}}{2} \delta(f - f_c) + \frac{e^{-j\phi}}{2} \delta(f + f_c) \]

\[ v_1(f) = S(f) * \left( \frac{e^{j\phi}}{2} e^{j2\pi f_c t} + \frac{e^{-j\phi}}{2} e^{-j2\pi f_c t} \right) \]
\[ = \frac{e^{j\phi}}{2} S(f) * \delta(f - f_c) + \frac{e^{-j\phi}}{2} S(f) * \delta(f + f_c) \]
\[ = \frac{e^{j\phi}}{2} S(f - f_c) + \frac{e^{-j\phi}}{2} S(f + f_c) \]
Costas Receiver: **In-Phase** Coherent Detector

Continuing . . .

\[ v_1(f) = \frac{e^{j\phi}}{2} S(f - f_c) + \frac{e^{-j\phi}}{2} S(f + f_c) \]

\[ = \frac{\cos(\phi) + j \sin(\phi)}{2} S(f - f_c) + \frac{\cos(-\phi) + j \sin(-\phi)}{2} S(f + f_c) \]

\[ = \frac{\cos(\phi) + j \sin(\phi)}{2} S(f - f_c) + \frac{\cos(\phi) - j \sin(\phi)}{2} S(f + f_c) \]

\[ = \frac{\cos(\phi)}{2} [S(f - f_c) + S(f + f_c)] + j \frac{\sin(\phi)}{2} [S(f - f_c) - S(f + f_c)] \]

\[ \approx \frac{1}{2} \quad \text{for } \phi \approx 0 \]

\[ v_1(f) \approx \frac{1}{2} [S(f - f_c) + S(f + f_c)] \]
Costas Receiver: In-Phase Coherent Detector

$S(f)$  DSB-SC Input

$S(f - f_c)$

$S(f + f_c)$

$[S(f - f_c) + S(f + f_c)]$

baseband components add coherently

$[S(f - f_c) - S(f + f_c)]$

baseband components cancel out
Costas Receiver: **In-Phase** Coherent Detector

\[
[S(f - f_c) + S(f + f_c)]
\]

-2\(f_c\) \(\rightarrow\) \(-f_c\) \(\rightarrow\) \(f_c\) \(\rightarrow\) \(2f_c\)

Baseband components add coherently.

\[
[S(f - f_c) - S(f + f_c)]
\]

-2\(f_c\) \(\rightarrow\) \(-f_c\) \(\rightarrow\) \(f_c\) \(\rightarrow\) \(2f_c\)

Baseband components cancel out.
Costas Receiver: **In-Phase** Coherent Detector

Summarizing the result:

\[ v_1(f) = \frac{e^{j\phi}}{2} S(f - f_c) + \frac{e^{-j\phi}}{2} S(f + f_c) \]

\[ = \frac{\cos(\phi) + j \sin(\phi)}{2} S(f - f_c) + \frac{\cos(-\phi) + j \sin(-\phi)}{2} S(f + f_c) \]

\[ = \frac{\cos(\phi) + j \sin(\phi)}{2} S(f - f_c) + \frac{\cos(\phi) - j \sin(\phi)}{2} S(f + f_c) \]

\[ = \frac{\cos(\phi)}{2} \left[ S(f - f_c) + S(f + f_c) \right] + \frac{\sin(\phi)}{2} \left[ S(f - f_c) - S(f + f_c) \right] \]

**significant baseband**  
**disappearing baseband as \( \phi \) approaches zero**

The closer \( \phi \) is to zero, the more significant the baseband term of \( v_1(t) \) and vice versa.
Costas Receiver: In-Phase Coherent Detector

For small $\phi$. 

DSB-SC Input

$|S(f)|$

$\frac{e^{j\phi}}{2} S(f - f_c)$

$\frac{e^{-j\phi}}{2} S(f + f_c)$

$\frac{e^{j\phi}}{2} S(f - f_c) + \frac{e^{-j\phi}}{2} S(f + f_c)$
Costas Receiver: Quadrature-Phase Detector

$A_C \cos(2\pi f_C t) \cdot \sin(2\pi f_C t + \phi) \cdot m(t)$

$= \frac{A_C}{2} \sin[2\pi(2 f_C t + \phi)] \cdot m(t) + \frac{A_C}{2} \sin(\phi) \cdot m(t)$

From $\cos(A)\sin(B) = \frac{1}{2}\sin(A + B) + \frac{1}{2}\sin(B - A)$
Costas Receiver: **Quadrature-Phase Detector**

Let 
\[ S(t) = A_C \cos(2\pi f_C t) \cdot m(t) \]
\[ v_Q(t) = S(t) \cdot \sin(2\pi f_C t + \phi) \]

Remember 
\[ \sin(2\pi f_C t + \phi) = \frac{e^{j\phi}}{2j} e^{j2\pi f_C t} - \frac{e^{-j\phi}}{2j} e^{-j2\pi f_C t} \]

**FT:** 
\[ \frac{e^{j\phi}}{2j} e^{j2\pi f_C t} - \frac{e^{-j\phi}}{2j} e^{-j2\pi f_C t} \Leftrightarrow \frac{e^{j\phi}}{2j} \delta(f - f_C) - \frac{e^{-j\phi}}{2j} \delta(f + f_C) \]

\[ v_I(f) = S(f) * \left( \frac{e^{j\phi}}{2j} e^{j2\pi f_C t} - \frac{e^{-j\phi}}{2j} e^{-j2\pi f_C t} \right) \]
\[ = \frac{e^{j\phi}}{2j} S(f) * \delta(f - f_C) - \frac{e^{-j\phi}}{2j} S(f) * \delta(f + f_C) \]
\[ = \frac{e^{j\phi}}{2j} S(f - f_C) - \frac{e^{-j\phi}}{2j} S(f + f_C) \]
Costas Receiver: Quadrature-Phase Detector

Summarizing the result:

\[ v_Q(f) = \frac{e^{j\phi}}{2j} S(f - f_c) - \frac{e^{-j\phi}}{2j} S(f + f_c) \]

\[ = \frac{\cos(\phi) + j\sin(\phi)}{2j} S(f - f_c) - \frac{\cos(-\phi) + j\sin(-\phi)}{2j} S(f + f_c) \]

\[ = \frac{\cos(\phi) + j\sin(\phi)}{2j} S(f - f_c) - \frac{\cos(\phi) - j\sin(\phi)}{2j} S(f + f_c) \]

\[ = \frac{\cos(\phi)}{2j} [S(f - f_c) - S(f + f_c)] + j \frac{\sin(\phi)}{2j} [S(f - f_c) + S(f + f_c)] \]

\[ \approx 0 \]

significant baseband

disappearing baseband as \( \phi \) approaches zero

The closer \( \phi \) is to zero, the more significant the baseband term of \( v_Q(t) \) and vice versa.
Costas Receiver: **Quadrature-Phase Detector**

\[
[S(f - f_c) - S(f + f_c)]
\]

- baseband components cancel out

\[
[S(f - f_c) + S(f + f_c)]
\]

- baseband components add coherently
Costas Receiver: Quadrature-Phase Detector

DSB-SC Input: \( |S(f)| \)

\[
\frac{e^{j\phi}}{2j} S(f - f_c)
\]

\[
\frac{e^{-j\phi}}{2j} S(f + f_c)
\]

\[
\frac{e^{j\phi}}{2j} S(f - f_c) - \frac{e^{-j\phi}}{2j} S(f + f_c)
\]

Small residual spectrum
Costas Receiver: $V_I(t)$ and $V_Q(t)$

\[
A_c \cos(2\pi f_c t) \cdot \cos(2\pi f_c t + \phi) \cdot m(t) = \frac{A_c}{2} \cos[2\pi(2f_c t + \phi)] \cdot m(t) + \frac{A_c}{2} \cos(\phi) \cdot m(t)
\]

\[
A_c \cos(2\pi f_c t) \cdot \sin(2\pi f_c t + \phi) \cdot m(t) = \frac{A_c}{2} \sin[2\pi(2f_c t + \phi)] \cdot m(t) + \frac{A_c}{2} \sin(\phi) \cdot m(t)
\]
Costas Receiver: $V_I(t)$ and $V_Q(t)$

\[
A_C \cos(2\pi f_c t) \cdot \cos(2\pi f_c t + \phi) \cdot m(t)
\]

\[
= \frac{A_C}{2} \cos[2\pi(2f_c) t + \phi] \cdot m(t) + \frac{A_C}{2} \cos(\phi) \cdot m(t)
\]

- DSB-SC signal input

- Product Modulator

- Low-Pass Filter

- Demodulated signal output

- Voltage-controlled oscillator

- Phase Discriminator

- Spectrum:

- Residual

Demodulated signal output

- $v_I(t)$

- $v_Q(t)$

- $v_0(t)$

Costas Receiver
Costas Receiver: Low-Pass Filtering

\[ A_c \cos(2\pi f_c t) \cdot \cos(2\pi f_c t + \phi) \cdot m(t) \]

\[ = \frac{A_c}{2} \cos(2\pi(2f_c)t + \phi) \cdot m(t) + \frac{A_c}{2} \cos(\phi) \cdot m(t) \]

Product Modulator \[ v_I(t) \]

\[ \cos(2\pi f_c t + \phi) \]

Voltage-controlled oscillator

-90 deg Phase Shifter

\[ \sin(2\pi f_c t + \phi) \]

Product Modulator \[ v_O(t) \]

Low-Pass Filter

Demodulated signal output \[ v_0(t) \]

Spectrum:

DSB-SC signal input

\[ A_c m(t) \cdot \cos(2\pi f_c t) \]

\[ = \frac{A_c}{2} \sin(2\pi(2f_c)t + \phi) \cdot m(t) + \frac{A_c}{2} \sin(\phi) \cdot m(t) \]

Costas Receiver
Costas Receiver: Local Oscillator Control

\[ A_c \cos(2\pi f_c t) \cdot \cos(2\pi f_c t + \phi) \cdot m(t) \]

\[ = \frac{A_c}{2} \cos[2\pi (2f_c) t + \phi] \cdot m(t) + \frac{A_c}{2} \cos(\phi) \cdot m(t) \]

Spectrum:

Demodulated signal output

\[ v_0(t) = \frac{A_c}{2} m(t) \cos(\phi) \]

\[ v'_0(t) = \frac{A_c}{2} m(t) \sin(\phi) \]

DSB-SC signal input

\[ A_c m(t) \cdot \cos(2\pi f_c t) \]

-90 deg Phase Shifter

\[ \cos(2\pi f_c t + \phi) \]

\[ \sin(2\pi f_c t + \phi) \]

Product Modulator

Voltage-controlled oscillator

Phase Discriminator

Product Modulator

Low-Pass Filter

Low-Pass Filter

Costas Receiver
**Costas Receiver: Phase Discriminator**

Phase difference is the error signal output of a phase discriminator. The error signal comes from applying \(v_0(t)\) and \(v'_0(t)\) to a multiplier. (Note: Phase lock implies phase coherence.)

\[
v_0(t) \cdot v'_0(t) = \frac{A_c}{2} m(t) \cos(\phi) \cdot \frac{A_c}{2} m(t) \sin(\phi)
\]

\[
= \left(\frac{A_c}{2}\right)^2 m^2(t) \cdot \cos(\phi) \cdot \sin(\phi)
\]

\[
\approx 1 \quad \approx \phi
\]

\[
= \left[\left(\frac{A_c}{2}\right)^2 m^2(t)\right] \phi \quad \text{for } \phi \ll 1
\]

Parameter \(\phi\) is the phase error signal fed to the oscillator.
Costas Receiver: Summary of Operation

\[ A_c m(t) \cdot \cos(2\pi f_c t) \]

DSB-SC signal input

\[ \cos(2\pi f_c t + \phi) \]
Local oscillator output

\[ \sin(2\pi f_c t + \phi) \]

Product Modulator

Voltage-controlled oscillator

Phase Discriminator

Circuit for phase locking \((\phi = 0)\)

Ideal phase lock at \(\phi = 0\)
Costas Receiver or Costas Loop