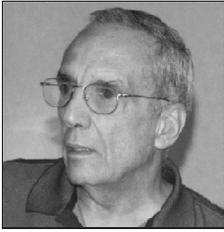# The Golden Ratio—A Contrary Viewpoint

Clement Falbo

Clement Falbo (clemfalbo@yahoo.com; Sonoma State University, Rohnert Park, CA 94928) was born in San Antonio, Texas, in 1931. He served in the U. S. Navy from 1951 to 1954. He attended the University of Texas in Austin, where he earned his B.A., M.A., and Ph.D. in mathematics. He taught college mathematics for 35 years, mostly at Sonoma State. After his retirement, he and his wife Jean joined the U.S. Peace Corps and taught high school in Zimbabwe for two years. Since then, they have been traveling widely.

Introduction

Over the past five centuries, a great deal of nonsense has been written about the golden ratio, $\Phi = \frac{1+\sqrt{5}}{2}$, its geometry, and the Fibonacci sequence. Many authors make claims that these mathematical entities are ubiquitous in nature, art, architecture, and anatomy. Gardner [**4**] has shown that the admiration for this number seems to have been raised to cult status. Fortunately, however, there have been some recent papers, including Fischler [**2**] in 1981, Markowsky [**7**] in 1992, Steinbach [**9**] in 1997, and Fowler [**3**] in 1982, that are beginning to set the record straight. For example, Markowsky, in his brilliant paper "Misconceptions about the Golden Ratio," speaking about Φ, says:

> "Generally, its mathematical properties are correctly stated, but much of what is presented about it in art, architecture, literature and esthetics is false or seriously misleading. Unfortunately, these statements about the golden ratio have achieved the status of common knowledge and are widely repeated. Even current high school geometry textbooks ... make many incorrect statements about the golden ratio. It would take a large book to document all the misinformation about the golden ratio, much of which is simply repetition of the same errors by different authors."

It is remarkable that prior to Fischler's and Markowsky's papers, there seemed to have been no set standards for obtaining measurements of artwork. Often, a proponent of the golden ratio will choose to frame some part of a work of art in an arbitrary way to create the appearance that the artist made use of an approximation of Φ. Markowsky shows an example in which Bergamini [**1**] arbitrarily circumscribes a golden rectangle about the figure of St. Jerome in a painting by Leonardo Da Vinci, cutting off the poor fellow's arm in order to make the picture fit.

It is frequently asserted that the golden ratio occurs in nature as the shape of spirals in sea shells. We can easily test this claim by first providing a protocol for measuring the spirals. One requirement should be to allow for some error in the measurements. Markowsky [**7**] suggests an error bar of $\pm 2\%$, which seems to be quite adequate. Measuring under this protocol, we find that spirals in sea shells do not generally fit the shape of the golden ratio. This is true despite the numerous articles on the Internet and elsewhere, in which pictures apparently have been stretched to fit the Φ ratio— "stretching the truth"—so to speak.

The golden ratio is associated with the Fibonacci sequence in a very simple way. The sequence is an example of a quadratic recursive equation sometimes used to describe various scientific and natural phenomena such as age-structured population growth. In order to define the general quadratic recursive formula, let x_0, x_1, p, and q be fixed positive numbers, and for any integer $n \geq 2$, define x_n as

$$x_n = px_{n-1} + qx_{n-2}. \tag{1}$$

Murthy [8] provides a number of theorems for this general recursive equation. It is clear that many of the features that are proclaimed to be unique to the Fibonacci sequence are, indeed, common to all second-order recursive equations. For example,

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = r, \tag{2}$$

where r is the positive root of the quadratic equation

$$x^2 - px - q = 0, \tag{3}$$

obtained by assuming that $x_n = x^n$ is a solution to (1). One of our objectives in this paper is to show that if $q = 1$ and r is the limit in (2), then the pair (r, p) has all of the geometric and algebraic properties that are often ascribed as being unique to the pair $(\Phi, 1)$. For example, we have $r - p = 1/r$ corresponding to the property $\Phi - 1 = 1/\Phi$.

For equation (1) to be useful in describing aged-structured population growth in plants and animals, the coefficients p and q must be determined by some niche or fecundity properties of the organism being studied. In other applications, such as phyllotaxy, we may use a second-order recursive equation such as (1) to predict and explain the evolution of leaf placement on a stem in terms of maximizing the gathering of sunlight. However, we should not expect the complexities of natural systems to yield to the easy-to-compute Fibonacci sequence, and there seems to be no unbiased evidence favoring the Fibonacci sequences over all other possible sequences. If one expends great effort in looking only for this special sequence, then it may be perceived, whether or not it is there. This is succinctly illustrated in terms of statistical analysis by Fischler [2], who shows that careless computations and misused formulas produce the golden number when it isn't there.

In a popular new book, Livio [6] draws upon the information developed in Markowsky and others to discuss the protocol violations that are the source of claims that Φ occurs in classical architecture, such as the Parthenon and the Egyptian pyramids. Livio presents a well-written explanation of misleading claims concerning these classics, as well as various paintings and other art work. Livio calls the advocates for these claims the "golden numberists." It seems, however, that he believes that certain constructions (such as Kepler's triangle) or arithmetic equations (such as a continued fraction representation of Φ) are significant and unique enough for him to subtitle his book "The Story of Phi, the World's Most Astonishing Number."

Origins of the golden ratio

The golden ratio is the solution to a problem given by Euclid (c. 300 BC) in his *Elements*, Book VI, Proposition 30:

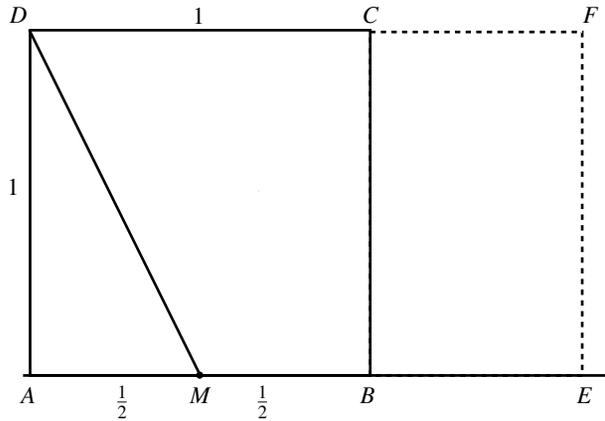*To cut a given finite line in extreme and mean ratio*.

Figure 1. The mean proportion $AE : AB :: AB : BE$

That is, given a segment $\overline{AE}$, find the point B for which $AE/AB = AB/BE$. Euclid had already solved this in a previous theorem (Book II, Proposition 11). We show his construction in Figure 1 in order to generalize it later.

Start with the unit square $ABCD$ and let M be the midpoint of $\overline{AB}$. Construct the line segment $\overline{MD}$. Draw a circle with center at M and radius $\overline{MD}$ so that it cuts $\overrightarrow{AB}$ at the point E. So, $MD = ME$. In modern notation, we have the lengths,

$$ME = \frac{\sqrt{5}}{2}, \quad MB = \frac{1}{2}, \quad \text{and} \quad BE = \frac{\sqrt{5}-1}{2}.$$

Now, since $AE = AB + BE$, or $AE = 1 + BE$, we can write: $AE = (1 + \sqrt{5})/2$. Thus, we can easily show that $AE/AB = AB/BE$.

The length AE, $(1 + \sqrt{5})/2$, is denoted by Φ, and is called the *golden ratio*, or the *divine proportion*. In the above figure, the rectangle $AEFD$ is called the *golden rectangle*. The history of the golden ratio pre-dates Euclid. As early as 540 BC, the Pythagoreans had studied it in their work with the pentagon. We discuss some of the ratios that appear in the pentagon and all other odd polygons later.

The golden ratio Φ can be used to construct a beautiful logarithmic spiral, shown in Figure 2. This graph can be obtained by fitting the polar equation $\rho = be^{c\theta}$ to se-
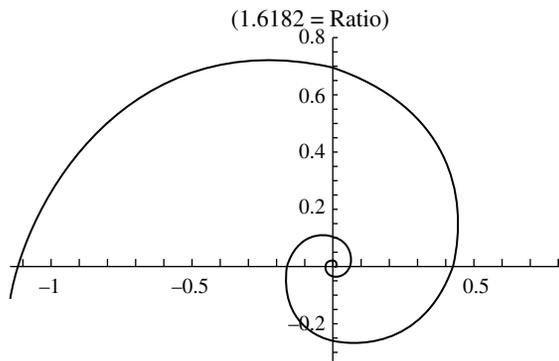

Figure 2. A spiral with the golden ratio

lected points on the golden rectangle. We can, however, also plot a logarithmic spiral inscribed in a rectangle with a ratio other than Φ. Figure 3 shows a spiral with a 1.33 to 1 ratio.
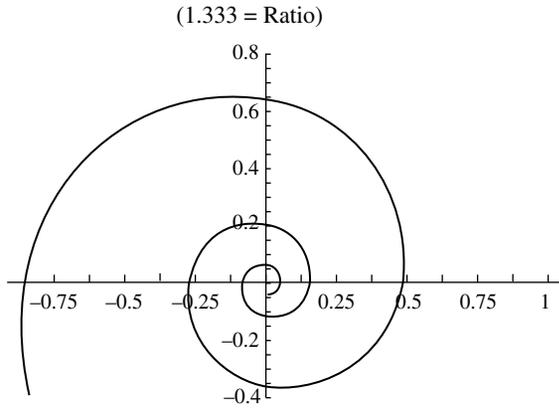

Figure 3. A spiral with a 1.33 to 1 ratio

Spirals in sea shells

Basically, the two types of sea shells that many of us are familiar with are the *cephalo-pod* (head-foot) and the *gastropod* (stomach-foot), or with apologies to my biologist friends, the octopus and the snail. The nautilus, a cephalopod, is an octopus in a shell that consists of a series of chambers, each sealed from the previous one. This animal lives in the latest chamber with its eight tentacles sticking out. In Figure 4 is a photo that I took of a longitudinal section of a nautilus.
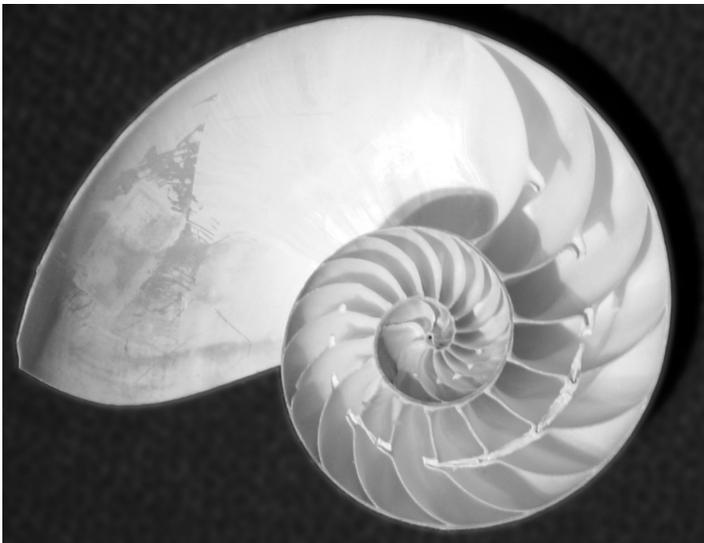

Figure 4. A digital photograph of a longitudinal section of a nautilus

The nautilus is definitely not in the shape of the golden ratio. Anyone with access to such a shell can see immediately that the ratio is somewhere around 4 to 3. In 1999, I measured shells of *Nautilus pompilius*, the chambered nautilus, in the collection at the California Academy of Sciences in San Francisco. The measurements were taken to the nearest millimeter, which gives them error bars of ± 1 mm. The ratios ranged from 1.24 to 1.43, and the average was 1.33, not Φ (which is approximately 1.618). Using Markowsky's $\pm 2\%$ allowance for Φ to be as small as 1.59, we see that 1.33 is quite far from this expanded value of Φ. It seems highly unlikely that there exists any nautilus shell that is within 2% of the golden ratio, and even if one were to be found, I think it would be rare rather than typical.

A ratio of different color, $\sqrt{2}$

A spiral of a different ratio is in the shape of the "silver ratio" $\sqrt{2}$ to 1. (Some authors call $1 + \sqrt{2}$ the silver ratio.) A spiral based on $\sqrt{2}$ is considerably closer to the shape of the nautilus than Φ is (see Figure 5).
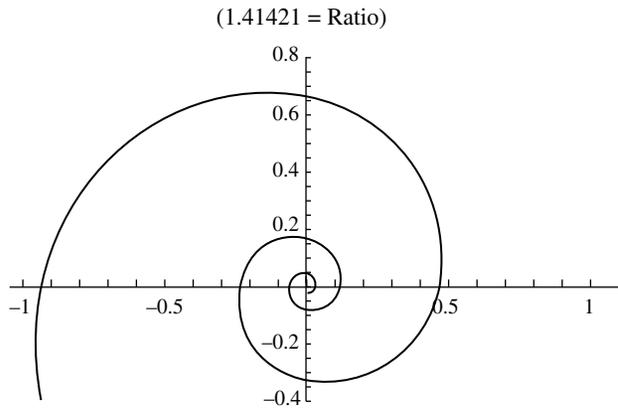

Figure 5. A spiral with the ratio $\sqrt{2}$ to 1

It is well known that the golden rectangle can be subdivided into subrectangles (each similar to the original) forming a kind of "spiral" of smaller and smaller similar rectangles. This interesting property happens to be shared by every rectangle (except the square). For example, if you take a $\sqrt{2}$-by-1 rectangle and cut it in two (in the natural way), you get two rectangles with sides in the same ratio. Continuing this, we get similar rectangles as shown in Figure 6.

(This rectangle was actually used in a field study in biology. In an article in *Science*, Harte, Kinzig, and Green [**5**] used Figure 6 when they wanted to determine a relation between species distribution and geographical area. They were interested in studying regions of varying sizes, but they wanted to maintain similarity in the shape of the region so as not to introduce a bias against rare species. One of their protocols was to start with a rectangle of known dimensions and then to generate a sequence of smaller similar subrectangles in which to collect samples. They chose the rectangle with the ratio of $\sqrt{2}$ to 1.)
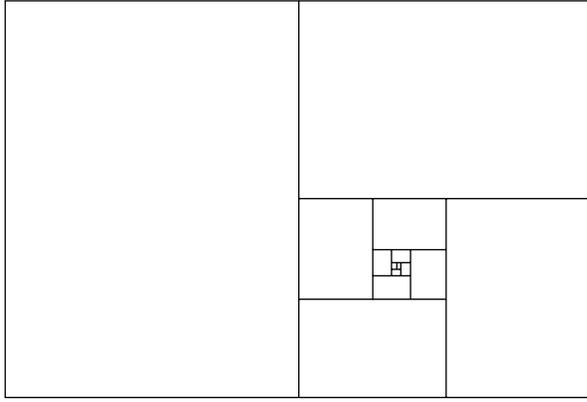
Figure 6. A rectangle and subrectangles with $\sqrt{2}$ to 1

The extreme mean ratio, generalized

Figure 6 illustrates the fact that the golden rectangle is not the only one that can be subdivided into similar subrectangles. Indeed, Fowler shows that we can construct an infinite sequence of extreme mean ratios. What he does is to generalize the standard procedure for constructing Φ. He starts with a unit square $ABCD$ and for any positive integer n, takes the point M_n on $\overrightarrow{AB}$, at a distance of $n/2$ from A. Then the segment $\overline{M_n D}$ has length $\sqrt{n^2 + 4}/2$. Now rotating $\overline{M_n D}$ about M_n he gets a circular arc that cuts $\overrightarrow{AB}$ at a point E. (M_n is between A and E.) The length of $\overline{AE}$ is $n + \sqrt{n^2 + 4}/2$, which he calls the *noem*, or *nth order extreme mean*. That is, $(M_n E + AM_n)/AB = AB/(M_n E - AM_n)$. If $n = 1$, the noem is Φ, or the first order extreme mean, while if $n = 2$, the noem is $1 + \sqrt{2}$. In Figure 7(a), $n = 3$, $AM_3 = \frac{3}{2}$, and G is a point so that M_3 is the midpoint of the segment $\overline{AG}$. $AE = (3 + \sqrt{13})/2$, (the third order extreme mean) and $GE = (-3 + \sqrt{13})/2$, so $AE/AB = AB/GE$. One property of the nth order extreme mean is that $noem - 1/noem = n$.
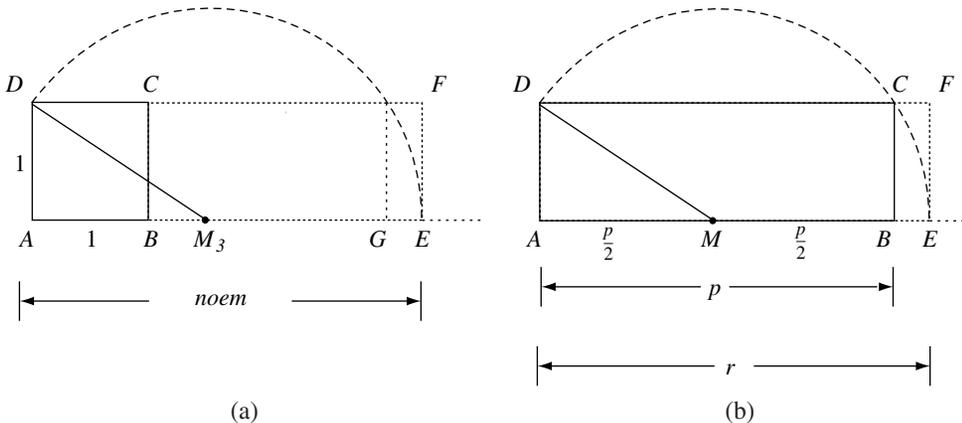

Figure 7. (a) The noem, with $n = 3$; (b) The poem, for any $p > 0$

We can obtain the same result for any positive real number p, not just for an integer n, as follows. After Fowler, we call our ratio the *pth order extreme mean*, or the *poem*. Its

construction is shown in Figure 7(b). Start with any p-by-1 rectangle $ABCD$, $AB = p$. Select the midpoint M of $\overline{AB}$ and construct the segment $\overline{MD}$; its length is $\sqrt{p^2 + 4}/2$. Rotate the radius $\overline{MD}$ about M until it cuts the line $\overrightarrow{AB}$ at the point E, giving the segment $\overline{AE}$ with length $(\sqrt{p^2 + 4} + p)/2$, denoted by r. Then $BE = AE - AB$; this length is $(-p + \sqrt{p^2 + 4})/2$, which is $1/r$. Thus, $r - p = 1/r$. The number r is the poem. Notice that r is a root of the quadratic equation (3) when $q = 1$. Completing the rectangle $BEFC$, as shown, we have that $AE/EF = FE/BE$.

Self-similar decompositions of rectangles of any ratio

The *poem*, defined above, has a simple interpretation. For any $r > 1$, and any rectangle $\mathcal{R}$ with a ratio of r to 1, we can divide $\mathcal{R}$ into two smaller rectangles, one of which has the same ratio r to 1, and this in turn can be so divided etc., as in Figure 8. The coordinates of the vertices A_4, A_5, A_6, are given below.

$$A_4 = \left(r - \frac{1}{r} + \frac{1}{r^3}, 0\right)$$

$$A_5 = \left(r - \frac{1}{r}, \frac{1}{r^2} - \frac{1}{r^4}\right)$$

$$A_6 = \left(r - \frac{1}{r} + \frac{1}{r^3} - \frac{1}{r^5}, \frac{1}{r^2}\right)$$

The vertices A_1, A_2, A_3, ... converge to a "center-point,"

$$\left(\frac{r^3}{1 + r^2}, \frac{1}{1 + r^2}\right).$$

In order to show this, we look at the abscissas of the corners. First notice that the corner points alternate in returning to a previous x value before moving on to a new x value. So, skipping those that return to a previous abscissa, we have:

Point	A_2	A_4	A_6	...
Abscissa	$r - \frac{1}{r}$	$r - \frac{1}{r} + \frac{1}{r^3}$	$r - \frac{1}{r} + \frac{1}{r^3} - \frac{1}{r^5}$	...

The alternating geometric series

$$r - \frac{1}{r} + \frac{1}{r^3} - \frac{1}{r^5} + \cdots \quad \text{converges to} \quad \frac{r^3}{1 + r^2}.$$

Similarly, the corner points return to a previous ordinate in an alternating pattern and they converge to $1/(1 + r^2)$. Notice that, for any r (including Φ, of course) the center point determined by these infinite series is the intersection of the line through $(0, 1)$ and $(r, 0)$ and the line through the points $(r - 1/r, 0)$ and $(r, 1)$.

Other geometric properties

Here are some other common geometric properties that are not unique to Φ.
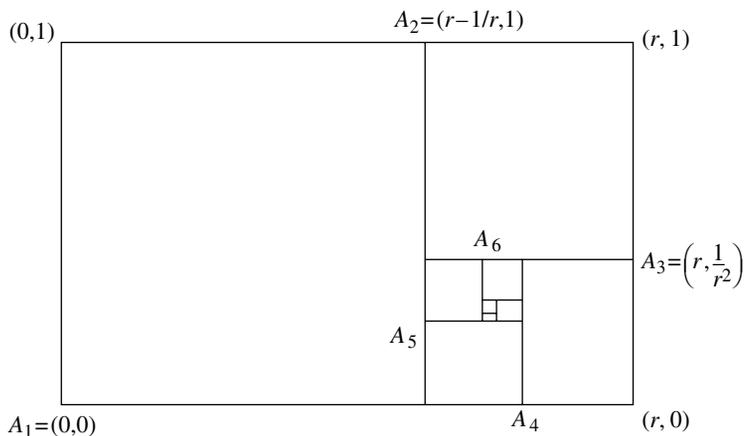
Figure 8. A rectangle of dimensions r to 1, with similar subrectangles

Spirals. An equiangular spiral can be drawn through the points $A_1, A_2, A_3, \ldots$ in Figure 8. Just use the center point

$$\left(\frac{r^3}{1+r^2}, \frac{1}{1+r^2} \right)$$

and fit a polar equation for the logarithmic spiral through any two of these points. It is a general property of all such spirals that the tangents to the spiral at any point make a fixed angle with the rays from the center point.

Sum of the areas of all of the subrectangles. If $r > 1$ and $p = r - 1/r$, then the sum of the areas,

$$r + \frac{1}{r} + \frac{1}{r^3} + \cdots,$$

of all the rectangles in Figure 8 is r^2/p, and in the special case in which $r = \Phi$, the sum is Φ^2 because $p = 1$.

Kepler's triangle. $\triangle ACD$ in Figure 9 is called Kepler's triangle; here is how it is constructed. Let $r > 1$ and again let $p = r - 1/r$; then $r^2 = rp + 1$. (As before, $p = 1$
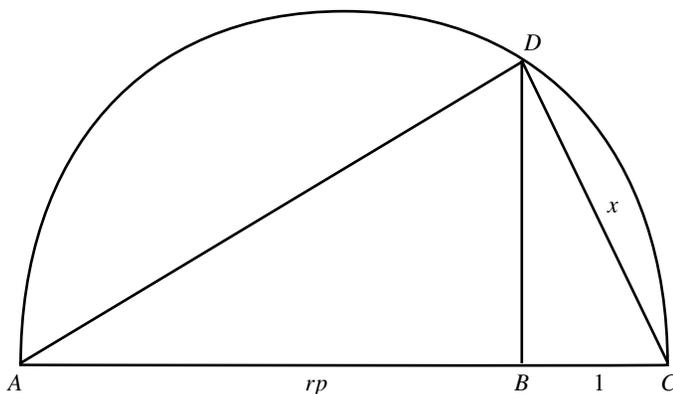

Figure 9. Kepler's Triangle

when $r = \Phi$.) Now construct a semicircle with $rp + 1$ as the diameter $\overline{AC}$. Mark the point B so that $AB = rp$ and $BC = 1$. Erect a line at B that intersects the circle at D. Then the length of $\overline{DC}$ (shown as x here) is r. Apparently this was thought to be amazing when Kepler did it, but it is not so amazing—it is simply the Euclidean construction for finding the square root of a segment $\overline{AB}$, that is $DB = \sqrt{rp}$. The hypotenuse of the right triangle DBC is $\sqrt{rp + 1} = r$.

Pentagons, heptagons, nonagons, ...

Regular polygons with an odd number of sides have some interesting properties related to the ratios we have been discussing. Steinbach makes the case for other polygons in his paper; he opens with this:

> "One of the best-kept secrets in plane geometry is the family of ratios of diagonals to sides in the regular polygons. So much attention has been given to one member of this family, the golden ratio Φ in the pentagonal case, that the others live in undeserved obscurity. But the wealth of material that pours from the pentagon—proportional sections, recursive sequences, and quasiperiodic systems—can be matched wonder-for-wonder by any other polygon . . . . "

It is well known that the golden ratio occurs in the regular pentagon with unit sides. In Figure 10, the length $AC = 2\cos(\pi/5)$, which is Φ.
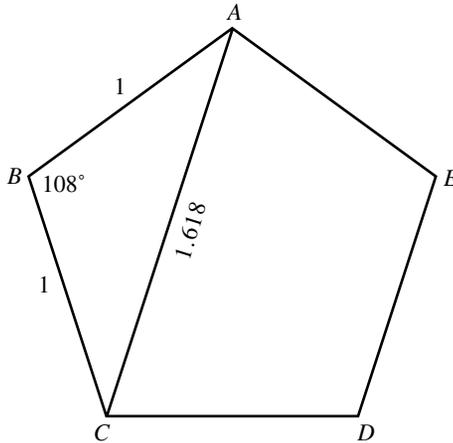

Figure 10. The pentagon and the golden ratio, $AC - AB = 1/AC$

Much of the publicity given to Φ has come from the special place that the pentagon had among the Pythagoreans. In this case, the equation $AC - AB = 1/AC$ says that $\Phi - 1 = 1/\Phi$. But, if we look at other regular odd polygons, we see that some pair of diagonals also satisfy the same relationship. Indeed, it is an interesting exercise to prove the following result.

Let $\mathcal{P}$ be any regular odd polygon with n sides, $n \geq 5$. If r is the length of a longest diagonal of $\mathcal{P}$ and p is the length of a second longest diagonal, then $r - p = 1/r$.
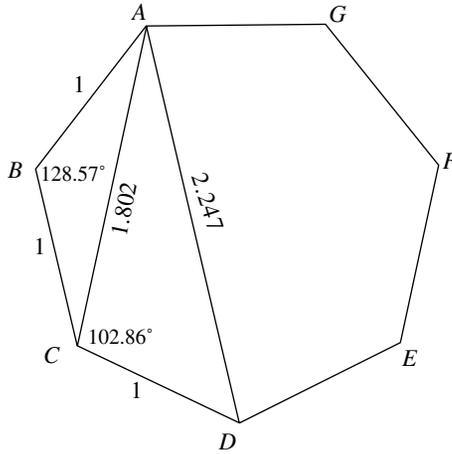
Figure 11. The heptagon and two related ratios. If $AD = r$, then $AC = r - 1/r$

Example. Behold the heptagon! See Figure 11. The diagonals $\overline{AD}$ and $\overline{AC}$ have the relationship described:

$$AD = 4\cos^2(\pi/7) - 1 \approx 2.247 \quad \text{and} \quad AC = 2\cos(\pi/7) \approx 1.802$$

That is, $AD - AC = 1/AD$.

Algebraic properties of $x_n = px_{n-1} + x_{n-2}$

We want to consider a modest generalization of the Fibonacci sequence, equation (4) below. The paper by Murthy presents substantially more general recursive equations that include (4).

Again, let $r > 1$ and $p = r - \frac{1}{r}$. Let $x_0 = 1$, $x_1 = 1$, and for $n \geq 2$, let

$$x_n = px_{n-1} + x_{n-2}. \tag{4}$$

Then the general term of the sequence is

$$x_n = c_1 r^n + c_2 \left(-\frac{1}{r}\right)^n, \tag{5}$$

where

$$r = \frac{p + \sqrt{p^2 + 4}}{2}, \qquad \frac{-1}{r} = \frac{p - \sqrt{p^2 + 4}}{2},$$

and

$$c_1 = \frac{2 - p + \sqrt{p^2 + 4}}{2\sqrt{p^2 + 4}}, \qquad c_2 = \frac{p - 2 + \sqrt{p^2 + 4}}{2\sqrt{p^2 + 4}}.$$

Because of the way that p is defined, r and $\frac{-1}{r}$ are the roots of the quadratic equation

$$x^2 - px - 1 = 0. \tag{6}$$

A few terms of this sequence, obtainable from either equation (4) or equation (5), are

$$x_2 = p + 1,$$
$$x_3 = p^2 + p + 1,$$
$$x_4 = p^3 + p^2 + 2p + 1.$$

If $p \neq 1$, this is not Fibonacci. We now point out several properties that any such solution r of (6) would have and that the quadratic equation is sufficient to imbue r with these properties.

Subtracting a number to get the reciprocal. From the fact that Φ is the solution to equation (6) when $p = 1$, $\Phi - 1 = 1/\Phi$. Thus, Φ is the only number whose reciprocal can be found by subtracting 1. But for any other positive number p, if r is the positive root of (6), then $r - p = 1/r$, and r is the only number whose reciprocal can be found by subtracting p. This is simply a relation between the coefficients and the sum of the roots of any quadratic equation. For example, if $p = \frac{1}{2}$, and r is the positive root of $x^2 - \frac{1}{2}x - 1 = 0$, then $r = (1 + \sqrt{17})/4$. Subtracting $\frac{1}{2}$ gives $(-1 + \sqrt{17})/4$, the reciprocal of $(1 + \sqrt{17})/4$.

Square of nth term. It is easy to prove by induction that for any three consecutive terms of the sequence in equation (4), the square of the middle term is p more or p less than the product of the other two terms. The Fibonacci special case says that given any three consecutive terms, the square of the middle one is 1 more or 1 less than the product of the other two.

Continued fraction and endless square root. Two other relations between r and p are

$$r = p + \cfrac{1}{p + \cfrac{1}{p + \cfrac{1}{p + \cdots}}} \quad \text{and} \quad r = \sqrt{1 + p\sqrt{1 + p\sqrt{1 + p \ldots}}},$$

which specialize when $p = 1$ to

$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \quad \text{and} \quad \Phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}.$$

A word in favor of Φ

We do recognize that the number Φ, like e and π, has many interesting properties. The sense in which this is true comes from the fact that Φ shows up when we try to simplify certain formulas, such as setting $p = 1$ in the continued fraction given above. This is the same sense in which the number e is interesting. If you want to find the derivative of $\log_b(x)$, $b > 0$ and $b \neq 1$, from the definition

$$\lim_{h \to 0} \frac{\log_b(x + h) - \log_b(x)}{h},$$

after some work, you get

$$\frac{1}{x} \log_b \left(\lim_{h \to 0} \left(1 + \frac{h}{x} \right)^{x/h} \right).$$

The indicated limit is e; so we have $\frac{1}{x} \log_b(e)$. Now, it is *natural* to select $b = e$; so, the desired derivative is $\frac{1}{x}$. Using e for the logarithm base just makes the formula simpler.

Conclusions

We have shown that in certain instances, the claim that the golden ratio has a special place among numbers as a valid description of nature is unsupported. Furthermore, it has been well refuted that this ratio is somehow exceptionally pleasing and that it occurs frequently in art and architecture. For example, from taking measurements, we find that there is no basis for saying that Φ is the ratio that naturally occurs in sea shells. In particular, there is no basis for the assertion that it occurs in the nautilus. I disagree with Livio's implication that Φ is "The World's Most Astonishing Number." The interesting properties of Φ as a positive root to the quadratic equation (6) (with $p = 1$) are matched by the positive root to (6) for any positive number, $p \neq 1$. The wonderful geometric properties of Φ as an extreme mean are matched by the other means discussed here. Also, finding Φ, as $2\cos(\pi/5)$, in the pentagon is exciting, but no more so than finding $4\cos^2(\pi/7) - 1$ in the heptagon. Perhaps, Φ deserves to be included in a list along with e, π, and other numbers because it can be used to simplify certain formulas. In that sense, it is interesting, perhaps even very interesting, but not entirely astonishing.

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