A PROOF OF THE MULLINEUX CONJECTURE

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Dedicated to the memory of Professor Brian Hartley

INTRODUCTION

A partition λ of a positive integer n is a sequence $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$ of integers such that $\sum \lambda_i = n$. For a positive integer p, a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ (or its Young diagram) is called p-regular if it does not have p or more equal parts, i.e. if there does not exist $t \leq m-(p-1)$ with $\lambda_t = \lambda_{t+1} = \cdots = \lambda_{t+p-1}$.

Let F be a field of characteristic $p > 0$. It is well known that irreducible representations of the symmetric group S_n over F are naturally parametrized by p-regular partitions of n (cf. for example [9, 12]). If λ is such a partition we denote the corresponding irreducible module by D^λ.

Let sgn_n be the one-dimensional sign representation of S_n over F; i.e., $\mathrm{sgn}_n \cong F$ as a vector space and $g \cdot f = \mathrm{sgn}(g) f$ for any $g \in S_n,\ f \in F$. Here $\mathrm{sgn}(g)$ is just the sign of the permutation g.

It is clear that for any irreducible D^λ, the tensor product $D^\lambda \otimes \mathrm{sgn}_n$ is also irreducible. The problem, usually called the problem of Mullineux, is to find the p-regular partition μ such that $D^\lambda \otimes \mathrm{sgn}_n \cong D^\mu$. Put

$$D^\lambda \otimes \mathrm{sgn}_n \cong D^{b_n(\lambda)}.$$

In this way a bijection b_n on the set P_n of p-regular partitions of n is defined for each positive integer n, and the problem is:

Problem 1. *Find b_n.*

The analogous question in characteristic zero is easily answered. The corresponding bijection on the set of *all* partitions of n is given by $\lambda \mapsto \lambda'$, where λ' is the conjugate partition to λ (the partition whose Young diagram is the transpose of the Young diagram of λ). So the hope is to find some sort of "conjugation modulo p".

The problem is important in the representation theory of the alternating group A_n (see [7]). In addition, the module $D^\lambda \otimes \mathrm{sgn}_n$ appears in a natural way in many situations. For example, if λ' is p-regular then:

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(1) the socle of the Specht module S^λ is $D^{\lambda'} \otimes \mathrm{sgn}_n$;

(2) the Young module Y^λ is the projective cover of $D^{\lambda'} \otimes \mathrm{sgn}_n$ (cf. [11]);

(3) the image of the Schur functor applied to the irreducible polynomial representation F_λ of $GL_n(F)$ is $D^{\lambda'} \otimes \mathrm{sgn}_n$ (cf. [8, Section 6]).

We refer the reader to [4, 5, 25] for other cases of the occurrence of b_n.

In 1979, Mullineux ([21]) proposed an algorithm which defines a bijection $m_n : P_n \to P_n$ (see Section 1 below for a detailed description) and conjectured that

Conjecture 2 (Mullineux)**.** $b_n = m_n$.

In a subsequent paper ([22]), Mullineux proved that for any (p-regular) λ, $b_n(\lambda)$ and $m_n(\lambda)$ have the same p-core. That result is used below.

Of the more recent results on the problem we would like to mention the following: Martin ([18, 19]) proved that $b_n = m_n$ if $n < 3p$. Andrews and Olsson [1] have shown that the numbers of fixed points of m_n and b_n coincide! Finally, Bessenrodt and Olsson ([2]) have proved that the Conjecture of Mullineux agrees with the (since proved) conjecture of Jantzen and Seitz (cf. [13, 14, 6, 15]).

Branching rules obtained by the second author made possible the approach we use here to prove Conjecture 2. In [16], the notion of a good node of a p-regular Young diagram is introduced (see Section 8 below) and it is proved that

$$\mathrm{soc}(D^\lambda \downarrow_{S_{n-1}}) \cong \bigoplus \{D^{\lambda_A} \mid A \text{ is good}\}$$

where λ_A means $\lambda \setminus \{A\}$ and $\mathrm{soc}(M)$ stands for the socle of the module M.

From an evident isomorphism

$$\mathrm{soc}((D^\lambda \otimes \mathrm{sgn}_n) \downarrow_{S_{n-1}}) \cong \mathrm{soc}(D^\lambda \downarrow_{S_{n-1}}) \otimes \mathrm{sgn}_{n-1}$$

we immediately get that for any good node A of λ there exists a good node B of $b_n(\lambda)$ such that

$$b_{n-1}(\lambda_A) = b_n(\lambda)_B.$$

So if Conjecture 2 is true, then this equation must hold for m in place of b. Thus Conjecture 2 implies the following.

Conjecture 3. *For any p-regular partition λ of any integer $n \geq 1$ there exist a good node A of λ and a good node B of $m_n(\lambda)$ such that $m_{n-1}(\lambda_A) = m_n(\lambda)_B$.*

The following result is proved in [17] and is of crucial importance.

Theorem 4. *Conjectures 2 and 3 are equivalent.*

Thus, representation theory is completely eliminated and we have to consider a purely combinatorial question about m_n.

We would like to mention that [17] actually contains a solution of Problem 1 different from that proposed by Mullineux. However, the Mullineux Conjecture seems to be important not only from a representation theoretic but also from a combinatorial point of view. Also, the Mullineux algorithm is more convenient to compute by hand. The two algorithms may prove to be useful in different situations. For example, the algorithm from [17] is convenient in inductive arguments, while the Mullineux construction allows one to easily determine the number of parts of the partition $b_n(\lambda)$.

Many questions about the combinatorial nature of both algorithms remain unclear, and we believe they will draw attention in the future.

The main result of the paper is:

Main Theorem. *Conjecture 2 holds.*

Actually, we prove Conjecture 3 and apply Theorem 4.

Remark. The classical Mullineux Conjecture proved here has a q-analog sometimes called the quantum Mullineux Conjecture (cf. [20, Section 7.6]). We are grateful to Gordon James for pointing out the following:

Let q be a root of unity in a field F, with $q \neq 1$ if the characteristic p of F is zero. Let e be the least positive integer such that $1+q+q^2+\cdots+q^{e-1}=0$, and let $\mathcal{H}_q$ be the Hecke algebra of type A over F with parameter q. If $\{T_w \mid w \in S_n\}$ is the usual basis of $\mathcal{H}_q$ then there exists an outer automorphism $^\#$ of $\mathcal{H}_q$, due to Goldman, given by $T_v^\# = (q-1)T_1 - T_v$, where v is any basic transposition in S_n. It is shown in [3] that irreducible $\mathcal{H}_q$-modules are parametrized by the e-regular partitions of n. If λ is such a partition, let D_q^λ be the corresponding irreducible module. One may define $(D_q^\lambda)^\#$ as the $\mathcal{H}_q$-module with the twisted action $h \circ v = \#(h)v$, for $h \in \mathcal{H}_q$ and $v \in D_q^\lambda$. Then $(D_q^\lambda)^\#$ is also irreducible, so $(D_q^\lambda)^\# \cong D_q^{\lambda^*}$ for some e-regular λ^*. The q-analog of the Mullineux Conjecture claims that $\lambda^* = m_n(\lambda)$ where m_n is the Mullineux map on the set of the e-regular partitions of n. Matthew Richards has proved in [24] that the permutation * on the e-regular partitions depends only on e, not on p and q. If $p>0$ and $q=1$ then $e=p$, $\mathcal{H}_q \cong FS_n$. So the classical Mullineux Conjecture implies the quantum version for prime e. For general e, the q-analog of the Mullineux Conjecture remains open.

For our purposes, the abacus notation for partitions (cf. [12]) turned out to be effective. In Section 1 we reformulate for abaci the notions introduced in [16] of removable, good, normal, etc. nodes, and determine what the Mullineux algorithm looks like in this setting. In Section 2 we investigate the effect of adding a p-edge to a partition on runner sizes of the corresponding abacus. Sections 3 and 4 present some lemmas on normal and degenerate beads appearing in the abacus, leading to the necessary and sufficient condition given in Section 5 for the occurence of a degnerate bead. In Sections 6 and 7 we finally compare λ and $m_n(\lambda)$, proving that a degenerate bead occurs in λ if and only if one occurs in $m_n(\lambda)$; this leads shortly to the proof of the Main Theorem, presented in Section 8.

The reader who is interested in the main structure of the proof rather than the details should begin with Section 8.

1. Preliminaries

In this section we introduce the notion of an abacus for a partition and explain the Mullineux procedure in this setting. More detailed information concerning the language of the abacus can be found in [12, Section 2.7] or [23].

Throughout the paper p is a fixed prime and $x \equiv y$ means $x \equiv y \pmod p$. If $a, b \in \mathbb{Z}$ we denote

$$[a,b] = \{c \in \mathbb{Z} \mid a \leq c \leq b\}, \qquad (a,b] = \{c \in \mathbb{Z} \mid a < c \leq b\},$$
$$[a,b) = \{c \in \mathbb{Z} \mid a \leq c < b\}, \qquad (a,b) = \{c \in \mathbb{Z} \mid a < c < b\}.$$

If $a \in \mathbb{Z}$ and $a = pq + r$, $q \in \mathbb{Z}$, $r \in (0, p]$, we write $\overline{a}$ for r.

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$ be a partition of n. We write $h(\lambda)$ for m (the *height* of λ). Gathering together equal parts of λ we write it in the form

$$\lambda = (\nu_1^{\alpha_1}, \dots, \nu_k^{\alpha_k}), \ \nu_1 > \cdots > \nu_k > 0, \ \alpha_i > 0. \tag{1}$$

Throughout the paper we assume that λ is p-regular, i.e. $\alpha_1, \dots, \alpha_k < p$. To explain the Mullineux algorithm we have to introduce some notation.

The *rim* of a Young diagram λ is its south-east border — in other words, a node in row i and column j of λ belongs to its rim if and only if the node in row $i+1$ and column $j+1$ does not belong to λ.

Example. Let $\lambda = (6, 4^2, 2, 1)$. Then the rim of λ contains the nodes represented by numbers in the following picture.

$$\begin{array}{llllll} \circ & \circ & \circ & 3 & 2 & 1 \\ \circ & \circ & \circ & 4 & & \\ \circ & 7 & 6 & 5 & & \\ 9 & 8 & & & & \\ 10 & & & & & \end{array}$$

Let us number the nodes of the rim moving from the "top-right" to the "left-bottom" (see the picture above). Define the first p-segment of the rim as the set consisting of the nodes whose numbers do not exceed p. If the last (i.e. having the largest number) node B of the first p-segment is in the last row of λ then λ has only one p-segment. If not, let r be the row containing B. The first node of the second p-segment is the node which has the smallest number, say δ, among the nodes of the rim lying in row $r+1$. The second p-segment is now defined as the set consisting of the nodes whose numbers i satisfy $\delta \leq i \leq \delta + p - 1$. Repeating this procedure sufficiently many times we reach the bottom row of the diagram. It is clear that all p-segments except possibly the last one contain p nodes. The *p-edge* is defined as the union of the p-segments.

Example. Let $\lambda = (6, 4^2, 2, 1)$, $p = 5$. The nodes of the p-edge (which consists of two p-segments) are coloured in black in the following picture.

$$\begin{array}{llllll} \circ & \circ & \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \circ & \bullet & & \\ \circ & \circ & \circ & \bullet & & \\ \bullet & \bullet & & & & \\ \bullet & & & & & \end{array}$$

Now define diagrams $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(z+1)}$ as follows. Put $\lambda^{(0)} = \lambda$, and for $i \geq 1$ put

$$\lambda^{(i)} = \lambda^{(i-1)} \setminus \{p\text{-edge of } \lambda^{(i-1)}\}.$$

We choose z to be maximal with respect to $\lambda^{(z)} \neq \emptyset$; so $\lambda^{(z+1)} = \emptyset$. We call the p-edge of $\lambda^{(j)}$ the j-th p-edge of λ. The *Mullineux symbol* for λ (introduced in [2]) is an array

$$G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}$$

where A_j is the number of nodes (or the length) of the j-th p-edge of λ, and $R_j = h(\lambda^{(j)})$ is the height of $\lambda^{(j)}$.

Example. Let $\lambda = (6,4^2,2,1)$, $p = 5$. The j-th p-edge contains the nodes labelled j in the following picture.

$$\begin{matrix} 3 & 2 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & & \\ 1 & 1 & 1 & 0 & & \\ 0 & 0 & & & & \\ 0 & & & & & \end{matrix}$$

Thus $\lambda^{(1)} = (3^3)$, $\lambda^{(2)} = (2^2)$, $\lambda^{(3)} = (1)$, $z = 3$, and

$$G_p(\lambda) = \begin{pmatrix} 8 & 5 & 3 & 1 \\ 5 & 3 & 2 & 1 \end{pmatrix}$$

For $i \in [0,z]$ put $\varepsilon_i = 0$ if $p|A_i$ and $\varepsilon_i = 1$ otherwise. The following proposition is proved in [21].

Proposition 1.1. *The entries of $G_p(\lambda)$ satisfy*

$$\begin{aligned} &\varepsilon_i \leq R_i - R_{i+1} < p + \varepsilon_i,\ 0 \leq i < z; \\ &1 \leq R_z < p + \varepsilon_z; \\ &R_i - R_{i+1} + \varepsilon_{i+1} \leq A_i - A_{i+1} < p + R_i - R_{i+1} + \varepsilon_{i+1},\ 0 \leq i < z; \\ &R_z \leq A_z < p + R_z; \\ &\sum_{i=0}^{z} A_i = n. \end{aligned}$$

Moreover, if $A_0, \dots, A_z, R_0, \dots, R_z$ are positive integers such that these inequalities are satisfied then there exists exactly one p-regular partition λ of n such that

$$G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}.$$

In fact, λ can easily be reconstructed from its Mullineux symbol by starting from the empty partition $\lambda^{(z+1)}$ and adding the j-th p-edges for $j = z, z-1, \dots, 0$. More precisely, if $j \in [0,z]$ and $\lambda^{(j+1)} = (l_1, \dots, l_x)$ has already been constructed, then to construct $\lambda^{(j)}$ one has to add the j-th p-edge which consists of A_j nodes in R_j rows. It is convenient to begin adding each p-edge from the bottom of the diagram. The following description is rather more confusing than the actual process; this will be clear if one simply tries to reconstruct λ from its Mullineux symbol for a few sample partitions λ.

If $R_j = R_{j+1}$ then $p|A_j$ by Proposition 1.1. Let $A_j = tp$; that is, we have to add t p-segments. We start adding from position $(x, l_x + 1)$ (i.e. the position in row x and column $(l_x + 1)$), and add p nodes one after another according to the following rule: Let B be the last added node. If the position above it is not occupied by a node of $\lambda^{(j+1)}$ then we add the next node at this position. Otherwise we add the next node at the position to the right of B. In this way we add the bottom p-segment of the j-th p-edge. Assume that we have already added $w < t$ segments from the bottom. Let u be the row in which the last node was added. Then we find the column v such that there is a node of $\lambda^{(j+1)}$ in position $(u-1, v-1)$ but none in position $(u-1, v)$. Add the first node of segment $w+1$ at position $(u-1, v)$ and add the remaining $p-1$ nodes of this segment following the rule described above.

If $R_j > R_{j+1}$ the only difference from the previous case is that we have to start adding the nodes of the bottom p-segment, which consists of $\overline{A_j} \in (0, p]$ nodes, from position $(R_j, 1)$.

Here is the bijection which Mullineux defined in [21]:

Definition 1.2. Let λ have Mullineux symbol

$$G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}.$$

Define a p-regular partition $m_n(\lambda)$ of n via

$$G_p(m_n(\lambda)) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ S_0 & S_1 & \dots & S_z \end{pmatrix}$$

where $S_j = A_j + \varepsilon_j - R_j$.

The proof of the fact that the symbol $\begin{pmatrix} A_0 & A_1 & \dots & A_z \\ S_0 & S_1 & \dots & S_z \end{pmatrix}$ does in fact correspond to a p-regular partition of n can be found in [21]; Proposition 1.1 ensures that $m_n(\lambda)$ is well defined.

We shall often write $m(\lambda)$ instead $m_l(\lambda)$, and we will use the evident equality $m(\lambda^{(i)}) = m(\lambda)^{(i)}$ without comment.

Example. Let $\lambda = (6, 4^2, 2, 1)$, $p = 5$. As we saw in the previous example,

$$G_p(\lambda) = \begin{pmatrix} 8 & 5 & 3 & 1 \\ 5 & 3 & 2 & 1 \end{pmatrix}.$$

So

$$G_p(m(\lambda)) = \begin{pmatrix} 8 & 5 & 3 & 1 \\ 4 & 2 & 2 & 1 \end{pmatrix}.$$

We reconstruct $m(\lambda)$ from its Mullineux symbol according to the rule described above:

$$\begin{matrix} 3 & 2 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & & \\ 0 & & & & & & \end{matrix}$$

Now we will describe the abacus notation. We consider sequences

$$\{s_0, s_1, \dots, s_v, \dots\}$$

where each s_v is either "$\cdot$" (space) or "$\times$" (bead) (one could call them 0 and 1 as well but, having in mind an abacus, we prefer this more representative terminology). For $\lambda = (\nu_1^{\alpha_1}, \dots, \nu_k^{\alpha_k})$ written in the form of (1) and $R \geq h(\lambda)$ we construct the sequence $s_R(\lambda)$ as follows: The first $R - h(\lambda)$ members of $s_R(\lambda)$ are beads. They are followed by ν_k spaces, then α_k beads, then $\nu_{k-1} - \nu_k$ spaces, then α_{k-1} beads, ..., then $\nu_1 - \nu_2$ spaces, then α_1 beads, and then spaces. (Informally speaking, we move along the rim of λ from bottom to top, putting beads in the sequence when we move up and leaving spaces when we move to the right).

Example. Let $\lambda = (6, 4^2, 2, 1)$. Then

$$s_5(\lambda) = \cdot \quad \times \quad \cdot \quad \times \quad \cdot \quad \cdot \quad \times \quad \times \quad \cdot \quad \cdot \quad \times,$$
$$s_6(\lambda) = \times \quad \cdot \quad \times \quad \cdot \quad \times \quad \cdot \quad \cdot \quad \times \quad \times \quad \cdot \quad \cdot \quad \times,$$
$$s_7(\lambda) = \times \quad \times \quad \cdot \quad \times \quad \cdot \quad \times \quad \cdot \quad \cdot \quad \times \quad \times \quad \cdot \quad \cdot \quad \times.$$

(we do not draw the spaces after the last bead).

Each sequence of spaces and beads with a finite number of beads is $s_R(\lambda)$ for exactly one λ and exactly one R, but there are infinitely many sequences corresponding to a fixed partition λ, namely $s_{h(\lambda)}, s_{h(\lambda)+1}, \dots$.

It is convenient to draw the elements $s_0, s_1, \dots$ in the array

$$\begin{array}{cccc} s_0 & s_1 & \dots & s_{p-1} \\ s_p & s_{p+1} & \dots & s_{2p-1} \\ s_{2p} & s_{2p+1} & \dots & s_{3p-1} \\ \vdots & \vdots & \dots & \vdots \end{array}$$

and for any residue $i \pmod p$ we say that the elements s_v with $v \equiv i$ form the i-th *runner* of the abacus. If s_v is a space we say that the v-th position of the abacus is unoccupied. We often denote beads by letters $k, l, m, \dots$. If $s_v = k$ we say that bead k occupies the v-th position of the abacus.

Warning. Denoting beads by letters is convenient, but it can be misleading, because we do not distinguish between abaci having the same bead configuration. For example, the abaci ($p = 5$)

$$\cdot \quad k \quad \cdot \quad l \quad \cdot$$

and

$$\cdot \quad l \quad \cdot \quad k \quad \cdot$$

are identified.

If the abacus is constructed from a sequence $s_R(\lambda)$ we say that it is an abacus for λ, and if it is constructed from $s_{h(\lambda)}(\lambda)$ we call it the *canonical* abacus for λ. One can easily see that the only abacus for λ with position 0 unoccupied is the canonical one.

Abaci for λ are in 1–1-correspondence with the so called sequences of β-numbers for λ (cf. [12, 10]). If $(\beta_1, \beta_2, \dots, \beta_r)$ is such a sequence then we obtain the corresponding abacus by placing beads in the abacus at positions $\beta_1, \beta_2, \dots, \beta_r$.

Example. Let $\lambda = (6, 4^2, 2, 1)$, $p = 5$. Then

$$\begin{matrix} \cdot & \times & \cdot & \times & \cdot \\ \cdot & \times & \times & \cdot & \cdot \\ \times & & & & \end{matrix}, \qquad \begin{matrix} \times & \cdot & \times & \cdot & \times \\ \cdot & \cdot & \times & \times & \cdot \\ \cdot & \times & & & \end{matrix}, \qquad \text{and} \qquad \begin{matrix} \times & \times & \cdot & \times & \cdot \\ \times & \cdot & \cdot & \times & \times \\ \cdot & \cdot & \times & & \end{matrix}$$

are abaci for λ, the first being canonical.

If it is clear which abacus is used for a partition λ, we shall often denote this abacus by the same letter λ. For a bead k of an abacus λ we denote by $P_\lambda(k)$ the position occupied by k in λ. If there is an unoccupied position $\alpha < P_\lambda(k)$ we call k *proper*; otherwise k is *improper*. All the beads of a canonical abacus are proper.

Now we want to describe the deletion and addition of a p-edge in the abacus notation. Informally speaking, if Λ is an abacus for $\lambda^{(j)}$ then to construct an abacus for $\lambda^{(j+1)}$ one should find the last bead of Λ and try to push it up one position. If this position is occupied by some proper bead l then the latter bead is pushed out by k and we try to move l one position up as we moved k. If the position above k is unoccupied then after moving k to this position we look for the bead m having the maximal position less than the *new* position of k and try to move m up similarly to k. Finally if k is in the top row of the abacus or the position above k is occupied by an improper bead then k moves to the smallest unoccupied position (this position will be in a different runner than that containing k in Λ) and the process of removing the p-edge is complete. We continue until the process completes. The procedure of adding p-edge is the reverse; we want to describe both operations formally.

Definition 1.3. Let Λ be an abacus for a nontrivial p-regular partition. We define the set $\{m_1, m_2, \dots, m_N\}$ of *r-movable* beads of Λ and the abacus $\varphi(\Lambda)$ as follows:

Let m_1 be the proper bead such that $P_\Lambda(m_1)$ is maximal (in other words, m_1 is the last bead of the abacus).

Assume that $m_1, \dots, m_i$ have been defined. If either

- $P_\Lambda(m_i) < p$ (i.e. m_i is in the first row of the abacus);
- $P_\Lambda(m_i) \geq p$ and $P_\Lambda(m_i) - p$ is occupied by an improper bead; or
- $P_\Lambda(m_i) \geq p$, $P_\Lambda(m_i) - p$ is unoccupied, and there does not exist a proper bead m with $P_\Lambda(m) < P_\Lambda(m_i) - p$;

then we put $N = i$. Otherwise, we define m_{i+1} to be the proper bead whose position is maximal with respect to the condition $P_\Lambda(m_{i+1}) \leq P_\Lambda(m_i) - p$.

Put

$$\alpha = \begin{cases} P_\Lambda(m_N) - p, & \text{if } P_\Lambda(m_N) \geq p \text{ and position } P_\Lambda(m_N) - p \text{ is unoccupied in } \Lambda; \\ \text{the minimal unoccupied position in } \Lambda, & \text{otherwise.} \end{cases}$$

Now we define $\varphi(\Lambda)$ to be the abacus with the same set of beads as Λ, but with $P_{\varphi(\Lambda)}(k) = P_\Lambda(k)$ for $k \neq m_1, m_2, \dots, m_N$, $P_{\varphi(\Lambda)}(m_N) = \alpha$, and $P_{\varphi(\Lambda)}(m_i) = P_\Lambda(m_i) - p$ for all $i \in [1, N)$. (In other words, to obtain $\varphi(\Lambda)$ from Λ, one has to move beads $m_1, \dots, m_{N-1}$ one position up, and move m_N to position α.)

Definition 1.4. Let λ be a p-regular partition with Mullineux symbol $G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}$ (possibly $z=-1$, which is interpreted as $\lambda = \emptyset$). Let A and R be positive integers, and let $\varepsilon = 0$ if $p|A$ and $\varepsilon = 1$ otherwise. Assume that

- $\varepsilon \le R - R_0 < p + \varepsilon$, $R - R_0 + \varepsilon_0 \le A - A_0 < p + R - R_0 + \varepsilon_0$, if $z \ge 0$;
- $1 \le R < p + \varepsilon$, $R \le A < p + R$, if $z = -1$.

Let Λ be an abacus for λ containing $Q \ge R$ beads. We define the set $n_1, \dots, n_D$ of *a-movable* beads and a new abacus $\psi(\Lambda) = \psi_{A,R}(\Lambda)$ as follows:

If $z \ge 0$ and $R = R_0$ then $p|A$, and we let n_1 be the first proper bead of Λ (i.e. the proper bead such that $P_\Lambda(n_1)$ is minimal), and put $P_{\psi(\Lambda)}(n_1) = P_\Lambda(n_1) + p = P_\Lambda(n_1) + \overline{A}$.

If $R > R_0$ or $z = -1$, we let n_1 be the (necessarily improper) bead of Λ with $P_\Lambda(n_1) = Q - R$, and put $P_{\psi(\Lambda)}(n_1) = P_\Lambda(n_1) + \overline{A}$.

Assume that $n_1, \dots, n_i$ and $P_{\psi(\Lambda)}(n_1), \dots, P_{\psi(\Lambda)}(n_i)$ have been defined. If $P_\Lambda(k) < P_{\psi(\Lambda)}(n_i)$ for all beads k of Λ then we put $D = i$. Otherwise, we define n_{i+1} to be the bead minimal with respect to $P_\Lambda(n_{i+1}) \ge P_{\psi(\Lambda)}(n_i)$, and define $P_{\psi(\Lambda)}(n_{i+1}) = P_\Lambda(n_{i+1}) + p$.

Finally, we define $\psi(\Lambda)$ as the abacus with the same set of beads as λ, but $P_{\psi(\Lambda)}(k) = P_\Lambda(k)$ if $k \ne n_1, \dots, n_D$ and $P_{\psi(\Lambda)}(n_i)$ defined as above. (In other words, to obtain $\psi(\Lambda)$ from Λ one should move n_1 to the position $P_\Lambda(n_1) + \overline{A}$, and beads $n_2, \dots, n_D$ down one row).

Part (i) of the following lemma is proved in [22, Section 3] (in the language of β-numbers), and (ii) follows from (i).

Lemma 1.5. *Let λ be a p-regular partition with Mullineux symbol*

$$G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}, \quad 0 \le j \le z.$$

i. *If $\Lambda^{(j)}$ is an abacus for $\lambda^{(j)}$, then*

$$\varphi(\Lambda^{(j)})$$

is an abacus for $\lambda^{(j+1)}$ (see Definition 1.3).

ii. *If $\Lambda^{(j+1)}$ is an abacus for $\lambda^{(j+1)}$ with at least R_j beads, then*

$$\psi_{A_j,R_j}(\Lambda^{(j+1)})$$

is an abacus for $\lambda^{(j)}$ (see Definition 1.4).

Example. Let $\lambda = (6, 4^2, 2, 1)$, $p = 5$. Let Λ be the canonical abacus for λ, i.e.

$$\Lambda = \begin{matrix} \cdot & a & \cdot & b & \cdot \\ \cdot & c & d & \cdot & \cdot \\ e & & & & \end{matrix}$$

(We use letters rather than $\times$ to denote the beads so that it will be easier to follow their movement). Then

$$\varphi(\Lambda) = \begin{array}{ccccc} b & a & \cdot & \cdot & \cdot \\ e & c & d & & \end{array} \quad \text{is an abacus for } \lambda^{(1)},$$

$$\varphi^2(\Lambda) = \begin{array}{ccccc} b & a & d & \cdot & \cdot \\ e & c & & & \end{array} \quad \text{is an abacus for } \lambda^{(2)},$$

$$\varphi^3(\Lambda) = \begin{array}{ccccc} b & a & d & c & \cdot \\ e & & & & \end{array} \quad \text{is an abacus for } \lambda^{(3)}, \text{ and}$$

$$\varphi^4(\Lambda) = \begin{array}{ccccc} b & a & d & c & e \end{array} \quad \text{is an abacus for } \lambda^{(4)} = \emptyset.$$

Recall from the example following Definition 1.2 that $m(\lambda)$ has Mullineux symbol

$$G_p(m_n(\lambda)) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ S_0 & S_1 & \dots & S_z \end{pmatrix}.$$

To reconstruct $m(\lambda)$ we should start with an abacus for $\lambda^{(4)} = \emptyset$ which has at least $S_0 = 4$ beads. So we begin with

$$M^{(4)} = \begin{array}{ccccc} r & s & t & u & \cdot \end{array}$$

Then

$$M^{(3)} = \psi_{1,1}(\Lambda^{(4)}) = \begin{array}{ccccc} r & s & t & \cdot & u \end{array}$$

$$M^{(2)} = \psi_{3,2}(\Lambda^{(3)}) = \begin{array}{ccccc} r & s & \cdot & \cdot & u \\ t & & & & \end{array}$$

$$M^{(1)} = \psi_{5,2}(\Lambda^{(2)}) = \begin{array}{ccccc} r & s & \cdot & \cdot & \cdot \\ t & \cdot & \cdot & \cdot & u \end{array}$$

$$M^{(0)} = \psi_{8,4}(\Lambda^{(1)}) = \begin{array}{ccccc} \cdot & s & \cdot & r & \cdot \\ \cdot & \cdot & \cdot & \cdot & u \\ t & & & & \end{array}$$

It is easily seen that $M^{(i)}$ is an abacus for $m(\lambda^{(i)}) = m(\lambda)^{(i)}$.

Definition 1.6. Let k be a proper bead of an abacus λ. Then k is called a *beginning* of λ if $P_\lambda(k) - 1$ is unoccupied in λ. It is called an *end* if $P_\lambda(k) + 1$ is unoccupied in λ. If k is a beginning we denote by $\lambda(k)$ the abacus obtained from λ by moving k to $P_\lambda(k) - 1$.

Definition 1.7. A bead k in runner a of an abacus λ is called *normal* if and only if

i. it is a beginning; and
ii. if $l_1, \dots, l_r$ are all the ends in runner $a-1$ of λ with $P_\lambda(l_v) > P_\lambda(k) - 1$ then there exist r distinct beginnings $k_1, \dots, k_r$ in runner a of λ such that $P_\lambda(l_v) + 1 > P_\lambda(k_v) > P_\lambda(k)$ for all $v \in [1, r]$.

A bead k of λ is called *good* if it is the top normal bead in its runner.

Remark. Ends and beginnings correspond to indent and removable nodes for the corresponding partition. Normal and good beads correspond to normal and good removable nodes, respectively. If the beginning k corresponds to a removable node A then $\lambda(k)$ is an abacus for λ_A (cf. [17] and Section 8).

Example. As we saw above, the canonical abacus for $(6,4^2,2,1)$ for $p=5$ is as follows:

$$\begin{matrix} \cdot & k & \cdot & l & \cdot \\ \cdot & m & q & \cdot & \cdot \\ t & & & & \end{matrix}$$

Here k,l,m,t are beginnings; k,l,q,t are ends. Of the beginnings, k is normal because for the end t there exists the beginning m as in 1.7; l is not normal because of the end q; m is not normal because of the end t; finally, t is normal. Both normal beads k and t turn out to be good in this example since they belong to distinct runners.

Definition 1.8. A good bead k of an abacus λ is called *degenerate* if and only if $\varphi(\lambda(k)) = \varphi(\lambda)$. Otherwise k is called non-degenerate.

In the example above, k is degenerate and t is non-degenerate.

Definition 1.9. If k and m are beads in the same runner of an abacus λ, then we write $k \succ m$ (in λ) if k is above m (i.e. $P_\lambda(k) < P_\lambda(m)$).

Notation. We denote by $(a)_\lambda$ the set of the *proper* beads in runner a of λ, and write $|a|_\lambda$ for the order of $(a)_\lambda$.

Definition 1.10. Let $L = \{l_1 \succ \cdots \succ l_r\} \subseteq (a-1)_\lambda$, $K = \{k_1 \succ \cdots \succ k_s\} \subseteq (a)_\lambda$. We say that K *compensates* L if $s \geq r$ and $P_\lambda(k_v) \leq P_\lambda(l_v) + 1$ for all $v \in [1,r]$. We say that L is *compensated* if there exists a subset $K \subseteq (a)_\lambda$ which compensates L.

The following lemma follows immediately from Definition 1.7.

Lemma 1.11. *A bead $k \in (a)_\lambda$ is normal (for λ) if and only if the set*

$$K = \{k' \in (a)_\lambda \mid k' \prec k\}$$

compensates the set

$$L = \{l \in (a-1)_\lambda \mid P_\lambda(l) \geq P_\lambda(k) - 1\}.$$

From now on we assume that λ is an arbitrary but fixed p-regular partition of n with Mullineux symbol

$$G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}$$

and we denote by the same letter λ the canonical abacus for λ (which contains R_0 beads). We denote by $\lambda^{(j)}$ the abacus $\varphi^j(\lambda)$ which is a (non-canonical in general) abacus for the partition $\lambda^{(j)}$, according to Lemma 1.5 ($j \in [0, z+1]$). Thus all abaci $\lambda^{(j)}$ contain R_0 beads. For example $\lambda^{(z+1)}$ is the abacus with beads in positions

$0,1,\dots,R_0-1$. We shall write $P_j(k)$, $|a|_j$, and $(a)_j$ for $P_{\lambda^{(j)}}(k)$, $|a|_{\lambda^{(j)}}$, and $(a)_{\lambda^{(j)}}$, respectively.

Similarly, $m(\lambda)$ is the canonical abacus for the partition $m(\lambda)$ (with S_0 beads), and we denote by $m(\lambda)^{(j)} = m(\lambda^{(j)})$ the abacus $\varphi^j(m(\lambda))$ which is a (non-canonical in general) abacus for the partition $m(\lambda)^{(j)} = m(\lambda^{(j)})$. All abaci $m(\lambda)^{(j)}$ have S_0 beads. We write $P^j(k)$, $|a|^j$, $(a)^j$, etc. for $P_{m(\lambda)^{(j)}}(k)$, $|a|_{m(\lambda)^{(j)}}$, $(a)_{m(\lambda)^{(j)}}$, etc.

Lemma 1.12. *Let $0 \le j \le z$.*

i. *Let l be a bead of $\lambda^{(j+1)}$ such that position $P_j(l)$ is unoccupied in $\lambda^{(j+1)}$. Then position $P_j(l)+1$ is unoccupied in $\lambda^{(j)}$.*
ii. *Let k be a bead of $\lambda^{(j)}$ which is proper for $\lambda^{(j+1)}$, and such that position $P_{j+1}(k)$ is unoccupied in $\lambda^{(j)}$. Then position $P_{j+1}(k)-1$ is unoccupied in $\lambda^{(j+1)}$.*

Proof. (i) Since $P_j(l)$ is unoccupied in $\lambda^{(j+1)}$, we have $P_j(l) \ne P_{j+1}(l)$, i.e. l is an a-movable bead of $\lambda^{(j+1)}$, or $l = n_w$ for some $w \in [1,D]$ (see 1.4). If $P_j(l)+1$ is unoccupied in $\lambda^{(j+1)}$ it will remain so in $\lambda^{(j)}$ because $P_j(n_v) < P_j(l)$ for $v < w$ and $P_{j+1}(n_v) \ge P_j(l)$ for $v > w$ by definition. If $P_j(l)+1$ is occupied in $\lambda^{(j+1)}$ then it is occupied necessarily by n_{w+1} and $P_j(n_{w+1}) > P_{j+1}(n_{w+1}) = P_j(l)+1$. So position $P_j(l)+1$ is unoccupied in $\lambda^{(j)}$ in this case, too. The proof of (ii) is similar. □

Definition 1.13. The subset $M = \{m_1 \succ \cdots \succ m_r\} \subseteq (b)_j$ is called *dense* (in $\lambda^{(j)}$) if $m_1 \succ m \succ m_r$ implies $m = m_v$ for some $v \in (1,r)$.

Lemma 1.14. *Let $L = \{l_1 \succ \cdots \succ l_r\} \subseteq (a-1)_{j+1}$, $K = \{k_1 \succ \cdots \succ k_s\} \subseteq (a)_{j+1}$. Assume that K is dense and compensates L in $\lambda^{(j+1)}$.*

i. *If there does not exist a bead k such that $P_j(k) = P_{j+1}(k_1)$, then K compensates L in $\lambda^{(j)}$.*
ii. *If there is a bead k such that $P_j(k) = P_{j+1}(k_1)$, then $K' = \{k, k_1, \dots, k_{r-1}\}$ compensates L in $\lambda^{(j)}$.*
iii. *If $P_{j+1}(l_1) > P_{j+1}(k_1)-1$, then K compensates L in $\lambda^{(j)}$.*

Proof. (i) and (ii). By definition $r \le s$ and $P_{j+1}(k_v) \le P_{j+1}(l_v)+1$, $v \in [1,r]$. If K does not compensate L in $\lambda^{(j)}$, choose w to be minimal such that $P_j(k_w) > P_j(l_w) + 1$. Then $P_{j+1}(k_w) = P_{j+1}(l_w)+1$, $P_j(l_w) = P_{j+1}(l_w)$, and $P_j(k_w) = P_{j+1}(k_w)+p$. By Lemma 1.12(ii) there exists a bead k with $P_j(k) = P_{j+1}(k_w)$ (k_w is pushed down by k). If $w > 1$ then, since K is dense, $k = k_{w-1}$. Since $P_j(l_{w-1}) < P_j(l_w) = P_j(k_{w-1})-1$, we have $P_j(k_{w-1}) > P_j(l_{w-1})+1$ which contradicts the choice of w. This proves (i). To prove (ii) notice that if $w = 1$ then we get $P_j(k) = P_j(l_1) + 1$, $P_j(k_v) \le P_j(l_{v+1})+1$, because $P_{j+1}(k_v) < P_{j+1}(l_{v+1})+1$ for $v \in [1,r)$.

(iii). Notice that $P_{j+1}(l_1) \ge P_{j+1}(k_1)-1+p$. Therefore $P_{j+1}(l_v) \ge P_{j+1}(k_1) - 1 + vp$ for $v \in [1,r]$. Let $y \ge 1$ be such that $P_{j+1}(k_v) = P_{j+1}(k_1)+(v-1)p$ for all $v \in [1,y]$, and $P_{j+1}(k_1)+yp$ is unoccupied in $\lambda^{(j+1)}$. Then $P_{j+1}(l_v) \ge P_{j+1}(k_v)-1+p$, hence $P_j(l_v) \ge P_j(k_v)-1$ for all $v \in [1,\min(r,y)]$, i.e. $\{k_1,\dots,k_{\min(r,y)}\}$ compensates $\{l_1,\dots,l_{\min(r,y)}\}$ in $\lambda^{(j)}$. If $r > y$, then $\{k_{y+1},\dots,k_r\}$ compensates $\{l_{y+1},\dots,l_r\}$ in $\lambda^{(j)}$ by part (i). □

Corollary 1.15. *Let $i \le j$ and let $L \subseteq (a-1)_j$ be a dense set which is compensated in $\lambda^{(j)}$. Then L is compensated in $\lambda^{(i)}$.*

Proof. This follows from Lemma 1.14, parts (i) and (ii). □

Lemma 1.16. *Let $L = \{l_1 \succ \cdots \succ l_r\} \subseteq (a-1)_j$, $K = \{k_1 \succ \cdots \succ k_s\} \subseteq (a)_{j+1} (\subseteq (a)_j)$. Assume that L is dense, there does not exist a bead l such that $P_{j+1}(l) = P_j(l_r)$, and K compensates L in $\lambda^{(j)}$. Then $L \subseteq (a-1)_{j+1}$ and K compensates L in $\lambda^{(j+1)}$.*

Proof. We first prove that $L \subseteq (a-1)_{j+1}$. If not, then $l_1 \notin (a-1)_{j+1}$. We have $P_j(k_1) \le P_j(l_1)+1$.

If $P_j(k_1) < P_j(l_1)+1$ then $P_j(k_1) \le P_j(l_1)+1-p$, and since $P_{j+1}(l_1) \ge P_j(l_1)-p$ we get $P_{j+1}(k_1) \le P_{j+1}(l_1)+1$. Now since k_1 is proper for $\lambda^{(j+1)}$ (by assumption), so is l_1; i.e. $l_1 \in (a-1)_{j+1}$.

If $P_j(k_1) = P_j(l_1)+1$, let $w \ge 1$ be minimal with respect to the conditions that $P_j(l_v) = P_j(l_1)+(v-1)p$, $v \in [1,w]$, and $P_j(l_1)+wp$ is unoccupied in $\lambda^{(j)}$. Then, since K compensates L in $\lambda^{(j)}$, one has $P_j(k_v) = P_j(l_v)+1$ for $v \in [1,w]$. Now it follows from Lemma 1.12(i) that $P_{j+1}(l_v) = P_j(l_v)$ for $v \in [1,w]$. In particular, $P_{j+1}(l_1) = P_j(l_1)$. Therefore $P_{j+1}(k_1) \le P_j(k_1) \le P_j(l_1)+1 = P_{j+1}(l_1)+1$, and then the fact that k_1 is proper for $\lambda^{(j+1)}$ implies l_1 is also.

Now let us prove that K compensates L in $\lambda^{(j)}$. If not, choose $w \le r$ to be maximal with respect to $P_{j+1}(l_w) < P_{j+1}(k_w)-1$. Then $P_j(l_w) = P_j(k_w)-1$, $P_{j+1}(l_w) = P_j(l_w)-p$, and $P_{j+1}(k_w) = P_j(k_w)$. Now it follows from Lemma 1.12(i) that there exists a bead l with $P_{j+1}(l) = P_j(l_w)$. By assumption, $w < r$. Then, in view of the density of L, $l = l_{w+1}$ and $P_{j+1}(l_{w+1}) = P_{j+1}(k_w)-1 < P_{j+1}(k_{w+1})-1$, which contradicts the choice of w. □

Definition 1.17. If k is a proper bead of λ we denote by $\mathrm{St}(k)$ ("St" for step) the integer j such that k is proper for $\lambda^{(j)}$ and is improper for $\lambda^{(j+1)}$. If $j = \mathrm{St}(k)$ we call k a *new* (proper) bead of $\lambda^{(j)}$.

Remark. We shall often use the following evident facts: If $k,m \in (a)_j$ and $k \succ m$ then $\mathrm{St}(k) \le \mathrm{St}(m)$. If $k,m \in (a)_j$ and $\mathrm{St}(k) < \mathrm{St}(m)$ then $k \succ m$.

Lemma 1.18. *A bead k is a new bead of $\lambda^{(j)}$ if and only if $R_0 - R_j \le P_{j+1}(k) \le R_0 - R_{j+1} - 1$.*

Proof. This is an immediate consequence of Definitions 1.17 and 1.4 and Lemma 1.5. □

The following constants measure the change from $\lambda^{(j+1)}$ to $\lambda^{(j)}$ (resp., from $m(\lambda)^{(j+1)}$ to $m(\lambda)^{(j)}$) in the difference between the sizes of runners a and $a-1$.

Definition 1.19. For all $j \in [0,z]$ define

$$d_j(a) = (|a|_j - |a|_{j+1}) - (|a-1|_j - |a-1|_{j+1}),$$

$$d^j(a) = (|a|^j - |a|^{j+1}) - (|a-1|^j - |a-1|^{j+1}).$$

2. RUNNER SIZES

In this section we investigate the changes that can occur in the size of a runner a, and in the difference of the sizes of two adjacent runners a and $a-1$, with the removal or addition of a p-edge to the abacus. We always assume that $0 \le j \le z$, and R_{z+1} is interpreted as 0.

Lemma 2.1. $|a|_{j+1} \le |a|_j \le |a|_{j+1} + 2$.

Proof. If $k \in (a)_{j+1}$ then $k \in (a)_j$. This implies the first inequality. If there are no improper beads in runner a of $\lambda^{(j+1)}$ then $|a|_j \le |a|_{j+1} + 1$ because at most one bead can move to runner a from another runner with the addition of the p-edge. Let m be the lowest (i.e. with $P_{j+1}(m)$ maximal) improper bead in runner a of $\lambda^{(j+1)}$. If m is the first row of $\lambda^{(j+1)}$, we obtain $|a|_j \le |a|_{j+1} + 2$ as above. If not, let l be the bead with $P_{j+1}(l) = P_{j+1}(m) - p$. It is enough to show that l is improper for $\lambda^{(j)}$. By 1.18, only the beads occupying positions from $R_0 - R_j$ to $R_0 - R_{j+1} - 1$ in $\lambda^{(j+1)}$ are new proper beads for $\lambda^{(j)}$. So l is improper for $\lambda^{(j)}$ in view of the inequality $R_j - R_{j+1} \le p$ given in Proposition 1.1. □

Lemma 2.2. $|a|_j = |a|_{j+1} + 2$ *if and only if* $A_j \not\equiv 0$, $a \equiv R_0 - R_j + A_j$, *and* $\overline{A_j} < R_j - R_{j+1}$. *Moreover, the two new proper beads in runner a of* $\lambda^{(j)}$ *are the beads m and k occupying the positions* $R_0 - R_j$ *and* $R_0 - R_j + \overline{A_j}$ *in* $\lambda^{(j+1)}$*, respectively, and* $P_j(m) = P_{j+1}(k)$, $P_j(k) = P_{j+1}(k) + p$. *(In other words, m moves from* $R_0 - R_j$ *to* $R_0 - R_j + \overline{A_j}$ *and pushes k down one row from this last position.)*

Proof. If $A_j \not\equiv 0$, $a \equiv R_0 - R_j + A_j$, and $\overline{A_j} < R_j - R_{j+1}$, let m be the (improper) bead occupying position $R_0 - R_j$ in $\lambda^{(j+1)}$. Since $\overline{A_j} < R_j - R_{j+1}$, position $R_0 - R_j + \overline{A_j}$ is occupied in $\lambda^{(j+1)}$ by some improper bead k (cf. 1.18. Notice that k is in runner a because $a \equiv R_0 - R_j + A_j$. When the j-th p-edge is added, m moves to position $R_0 - R_j + \overline{A_j}$ and pushes k out, to position $R_0 - R_j + \overline{A_j} + p$. Notice that k and m are proper for $\lambda^{(j)}$ because $P_{j+1}(m)$ is unoccupied in $\lambda^{(j)}$. Hence $|a|_j \ge |a|_{j+1} + 2$. So Lemma 2.1 implies $|a|_j = |a|_{j+1} + 2$.

In the other direction, let $|a|_j = |a|_{j+1} + 2$. Then there are some improper beads in runner a of $\lambda^{(j+1)}$ because otherwise $|a|_j \le |a|_{j+1} + 1$ (recall that at most one bead changes runners with the addition of a p-edge). Let k be the lowest improper bead in this runner. Then $|a|_j = |a|_{j+1} + 2$ implies that k becomes proper for $\lambda^{(j)}$ and some new bead comes to runner a from another runner. Therefore $A_j \not\equiv 0$, $a \equiv R_0 - R_j + A_j$, and $\overline{A_j} < R_j - R_{j+1}$. □

Corollary 2.3. *If* $|a|_j = |a|_{j+1} + 2$ *then* $|b|_j \le |b|_{j+1} + 1$ *for any* $b \not\equiv a$.

Proof. This follows from the Lemmas 2.2, 2.1. □

Lemma 2.4. $|a|_j = |a|_{j+1} + 1$ *if and only if one of the following conditions holds.*

i. $a \equiv R_0 - R_j + x$ *with* $0 < x < R_j - R_{j+1}$, $x \not\equiv A_j$. *In this case the bead occupying position* $R_0 - R_j + x$ *in* $\lambda^{(j+1)}$ *remains at the same position and becomes a new proper bead in runner a of* $\lambda^{(j)}$.

ii. $a \equiv R_0 - R_j + A_j$, $\overline{A_j} \geq R_j - R_{j+1} > 0$. *In this case the bead occupying position $R_0 - R_j$ in $\lambda^{(j+1)}$ moves to position $R_0 - R_j + \overline{A_j}$ and becomes a new proper bead in runner a of $\lambda^{(j)}$.*

Proof. The "if" direction is straightforward. For example, if $a \equiv R_0 - R_j + A_j$ and $\overline{A_j} \geq R_j - R_{j+1} > 0$, then the bead k occupying position $R_0 - R_j$ in $\lambda^{(j+1)}$ moves to position $R_0 - R_j + \overline{A_j}$ in runner a with the addition of the j-th p-edge (if $A_j \equiv 0$ we need to use $R_j - R_{j+1} > 0$). Moreover, k is improper for $\lambda^{(j+1)}$ and proper for $\lambda^{(j)}$. It follows from $\overline{A_j} \geq R_j - R_{j+1}$ and Lemma 1.18 that k is the only new proper bead in runner a of $\lambda^{(j)}$.

Let us prove "only-if". First, note that $R_j - R_{j+1} > 0$ because some new proper beads must appear.

If $A_j \equiv 0$, then by Lemma 1.18 the new proper beads of $\lambda^{(j)}$ are the bead k which has moved from position $R_0 - R_j$ in $\lambda^{(j+1)}$ to position $R_0 - R_j + p$ in $\lambda^{(j)}$, and the beads occupying positions $R_0 - R_j + 1$ through $R_0 - R_{j+1} - 1$ in both $\lambda^{(j+1)}$ and $\lambda^{(j)}$. So since $|a|_j > |a|_{j+1}$, we must have $a \equiv R_0 - R_j + x$ for some $x \in [0, R_j - R_{j+1})$. If $a \equiv R_0 - R_j$ we are in case (ii), as $A_j \equiv 0$; if $a \equiv R_0 - R_j + x$ with $0 < x < R_j - R_{j+1}$ we are in case (i).

If $A_j \not\equiv 0$ and $\overline{A_j} \geq R_j - R_{j+1}$, then by Lemma 1.18 the new proper beads of $\lambda^{(j)}$ are the bead k which has moved from position $R_0 - R_j$ in $\lambda^{(j+1)}$ to position $R_0 - R_j + \overline{A_j}$ in $\lambda^{(j)}$, and the beads occupying positions $R_0 - R_j + 1$ through $R_0 - R_{j+1} - 1$ in both $\lambda^{(j+1)}$ and $\lambda^{(j)}$. Again, since $|a|_j > |a|_{j+1}$, we must have $a \equiv R_0 - R_j + \overline{A_j}$ or $a \equiv R_0 - R_j + x$ for $x \in (0, R_j - R_{j+1})$. If $a \equiv R_0 - R_j + \overline{A_j}$, we have (ii); and if $a \equiv R_0 - R_j + x$ for $x \in (0, R_j - R_{j+1})$, we have (i).

Finally if $A_j \not\equiv 0$ and $\overline{A_j} < R_j - R_{j+1}$ we find in a similar manner that we are in case (i). □

Lemma 2.5. $-2 \leq d_j(a) \leq 2$.

Proof. This follows from Lemma 2.1 (and Definition 1.19). □

Lemma 2.6. $d_j(a) = 2$ *if and only if* $A_j \equiv 1$, $a \equiv R_0 - R_j + 1$, *and* $R_j - R_{j+1} > 1$. *Moreover, in this case the bead m which occupies position $R_0 - R_j$ in $\lambda^{(j+1)}$ moves to position $R_0 - R_j + 1$ and pushes down by one row the bead k which occupies $R_0 - R_j + 1$ in $\lambda^{(j+1)}$. Beads k and m are the two new proper beads in runner a of $\lambda^{(j)}$, and runner $a-1$ of $\lambda^{(j)}$ does not have new proper beads.*

Proof. By Lemma 2.1, $d_j(a) = 2$ if and only if $|a|_j = |a|_{j+1} + 2$, $|a-1|_j = |a-1|_{j+1}$. According to Lemma 2.2, the first equation holds if and only if $A_j \not\equiv 0$, $a \equiv R_0 - R_j + A_j$, $R_j - R_{j+1} > \overline{A_j}$. Now if $A_j \not\equiv 1$, it follows from Lemma 2.4(i) that $|a-1|_j = |a-1|_{j+1} + 1$. If $A_j \equiv 1$, it follows from 2.4, 2.2, and 2.1 that $|a-1|_j = |a-1|_{j+1}$. Thus, the first part of the Lemma is proved. The second part follows from the second part of Lemma 2.2. □

Lemma 2.7. $d_j(a) = 1$ *if and only if one of the following conditions holds.*

i. $a \equiv R_0 - R_j + A_j$; $R_j - R_{j+1} > 0$; $\overline{A_j} > R_j - R_{j+1}$ *if* $R_j - R_{j+1} > 1$ *and* $\overline{A_j} \geq 1$ *if* $R_j - R_{j+1} = 1$. *In this case the bead k occupying position $R_0 - R_j$ in $\lambda^{(j+1)}$*

moves to position $R_0 - R_j + \overline{A_j}$ and becomes the new proper bead in runner a of $\lambda^{(j)}$. Runner $a-1$ of $\lambda^{(j)}$ does not have new proper beads.

ii. *$\overline{A_j} \not\equiv 0, 1$, $a \equiv R_0 - R_j + A_j$, and $\overline{A_j} < R_j - R_{j+1}$. In this case the bead m occupying position $R_0 - R_j$ in $\lambda^{(j+1)}$ moves to $R_0 - R_j + \overline{A_j}$ and pushes a bead k down one row from the latter position. Beads k and m are the new proper beads in runner a of $\lambda^{(j)}$, and the bead l occupying $R_0 - R_j + \overline{A_j} - 1$ in both $\lambda^{(j+1)}$ and $\lambda^{(j)}$ is the new proper bead in runner $a-1$ of $\lambda^{(j)}$.*

iii. *$\overline{A_j} \not\equiv 0, 1$, $a \equiv R_0 - R_j + 1$, and $R_j - R_{j+1} > 1$. In this case the bead occupying $R_0 - R_j + 1$ in both $\lambda^{(j+1)}$ and $\lambda^{(j)}$ becomes the new proper bead in runner a of $\lambda^{(j)}$. Runner $a-1$ of $\lambda^{(j)}$ does not have new proper beads.*

Proof. By Lemma 2.1, $d_j(a) = 1$ if and only if one of the following conditions holds.

a. $|a|_j = |a|_{j+1} + 2$, $|a-1|_j = |a-1|_{j+1} + 1$;
b. $|a|_j = |a|_{j+1} + 1$, $|a-1|_j = |a-1|_{j+1}$.

By 2.2 and 2.4, (a) holds if and only if $A_j \not\equiv 0, 1$, $a \equiv R_0 - R_j + A_j$, and $\overline{A_j} < R_j - R_{j+1}$; thus we are in case (ii).

By 2.4, $|a|_j = |a|_{j+1} + 1$ if and only if one of the following conditions holds.

(α) $a \equiv R_0 - R_j + x$, $0 < x < R_j - R_{j+1}$, $x \not\equiv A_j$.
(β) $a \equiv R_0 - R_j + A_j$, $\overline{A_j} \geq R_j - R_{j+1} > 0$.

If (α) holds, then $|a-1|_j = |a-1|_{j+1}$ if and only if $x \equiv 1$ and $A_j \not\equiv 0$, in view of 2.2 and 2.4. Thus we are in case (iii).

If (β) holds, then $|a-1|_j = |a-1|_{j+1}$ if and only if $\overline{A_j} > R_j - R_{j+1}$ for $R_j - R_{j+1} > 1$ and $\overline{A_j} \geq 1$ for $R_j - R_{j+1} > 1$, according to 2.2 and 2.4. This gives (i). □

Lemma 2.8. *$d_j(a) = -1$ if and only if one of the following conditions holds.*

i. *$A_j \equiv 0$, $R_j - R_{j+1} > 0$, and $a \equiv R_0 - R_{j+1}$;*
ii. *$A_j \not\equiv 0$, $A_j \not\equiv R_j - R_{j+1}$, $A_j \not\equiv R_j - R_{j+1} - 1$, and $a \equiv R_0 - R_{j+1}$;*
iii. *$A_j \not\equiv 0$, $A_j \not\equiv R_j - R_{j+1} - 1$, and $a \equiv R_0 - R_j + A_j + 1$.*

Moreover, in cases (i) *and* (ii)*, runner a of $\lambda^{(j)}$ does not have new proper beads, and the bead l occupying position $R_0 - R_{j+1} - 1$ in $\lambda^{(j+1)}$ is the new proper bead in runner $a-1$ of $\lambda^{(j)}$. In case* (iii) *the bead l occupying $R_0 - R_j$ in $\lambda^{(j+1)}$ moves to position $R_0 - R_j + \overline{A_j}$ in runner $a-1$ of $\lambda^{(j)}$. If $\overline{A_j} > R_j - R_{j+1} - 1$ then l is the only new proper bead in runner $a-1$ of $\lambda^{(j)}$, and runner a of $\lambda^{(j)}$ has no new proper beads. If $\overline{A_j} < R_j - R_{j+1} - 1$ then l pushes down a bead m occupying $R_0 - R_j + \overline{A_j}$ in $\lambda^{(j+1)}$, and the beads l and m are the new proper beads in runner $a-1$ of $\lambda^{(j)}$; while the bead k occupying $R_0 - R_j + \overline{A_j} + 1$ in both $\lambda^{(j)}$ and $\lambda^{(j+1)}$ is the new proper bead in runner a of $\lambda^{(j)}$.*

Proof. This follows from from 2.1, 2.2, and 2.4 in a fashion similar to the proofs of Lemmas 2.6 and 2.7. □

Lemma 2.9. *$d_j(a) = -2$ if and only if $A_j \not\equiv 0$, $A_j \equiv R_j - R_{j+1} - 1$, and $a \equiv R_0 - R_{j+1}$. In this case the bead l occupying position $R_0 - R_j$ in $\lambda^{(j+1)}$ moves to position $R_0 - R_{j+1} - 1$ (in runner $a-1$) occupied by an improper bead m, and pushes m*

down one row. Beads m and l are the two new proper beads in runner $a-1$ of $\lambda^{(j)}$. Runner a of $\lambda^{(j)}$ has no new proper beads.

Proof. Again, this follows from from 2.1, 2.2, and 2.4 in a fashion similar to the proofs of Lemmas 2.6 and 2.7. □

3. SOME LEMMAS ON NORMAL BEADS

Throughout the section $0 \leq j \leq z$.

Lemma 3.1. *Let k be a normal bead for $\lambda^{(j)}$. Assume that either k is proper for $\lambda^{(j+1)}$ or $A_j \equiv 0$. Then position $P_j(k)-1$ is unoccupied in $\lambda^{(j+1)}$.*

Proof. Assume on the contrary that l_1 is the bead with $P_{j+1}(l_1) = P_j(k)-1$. Then since k is a beginning in $\lambda^{(j)}$, $P_j(l_1) \neq P_j(k)-1$. But since either k is proper for $\lambda^{(j+1)}$ or $A_j \equiv 0$, the only way that l_1 can move is one row up. So we have $P_j(l_1) = P_j(k)-1+p$. Let $l_1,\ldots,l_w$ be the beads such that $P_j(l_v) = P_j(k)-1+vp$ for $v \in [1,w]$ and $P_j(k)-1+(w+1)p$ is unoccupied in $\lambda^{(j)}$. Since k is normal, there exist beads $k_1,\ldots,k_w$ with $P_j(k_v) = P_j(k)+vp$, $v \in [1,w]$. Therefore $P_{j+1}(l_v) = P_j(l_v)$ for $v \in [1,w]$ by 1.12(i). Taking $v=1$ gives a contradiction. □

Lemma 3.2. *Let $k \in (a)_{j+1}$ be a normal bead for $\lambda^{(j+1)}$, and let $y \geq 0$ be the minimal integer such that position $P_{j+1}(k)-1+(y+1)p$ is unoccupied in $\lambda^{(j+1)}$. Then there are beads k_v with $P_{j+1}(k_v) = P_{j+1}(k)+vp$, $v \in [1,y]$. Moreover,*

- i. *Assume that k is a beginning for $\lambda^{(j)}$. Then k is normal for $\lambda^{(j)}$, and if k is not good for $\lambda^{(j+1)}$ then it is not good for $\lambda^{(j)}$.*
- ii. *Assume that k is not a beginning for $\lambda^{(j)}$. Then $y \geq 1$, k_y is normal for $\lambda^{(j)}$ and if k is not good for $\lambda^{(j+1)}$ then k_y is not good for $\lambda^{(j)}$.*

Proof. Let

$$L = \{l_1 \succ \cdots \succ l_r\} = \{l \in (a-1)_{j+1} \mid P_{j+1}(l) \geq P_{j+1}(k)-1\},$$
$$K = \{k_1 \succ \cdots \succ k_s\} = \{k' \in (a)_{j+1} \mid k' \prec k\}.$$

Then according to Lemma 1.11, K compensates L in $\lambda^{(j+1)}$. By assumption, for $v \in [1,y]$ we have $P_{j+1}(l_v) = P_{j+1}(k)-1+vp$. So $P_{j+1}(k_v) = P_{j+1}(k)+vp$ for $v \in [1,y]$.

Let x be minimal with respect to $P_{j+1}(k_v) = P_{j+1}(k)+vp$ for all $v \in [1,x]$ and $P_{j+1}(k)+(x+1)p$ is unoccupied in $\lambda^{(j+1)}$. Then $x \geq y$.

Assume that k is a beginning for $\lambda^{(j)}$. This means that either $P_j(k) = P_{j+1}(k)$ or $P_j(k) = P_{j+1}(k)+p$ and $y=0$. In both cases we have

$$\{l \in (a-1)_j \mid P_j(l) \geq P_j(k)-1\} = L,$$
$$\{k' \in (a)_j \mid k' \prec k\} = K.$$

So in view of Lemma 1.11, the k is normal for $\lambda^{(j)}$ if and only if K compensates L in $\lambda^{(j)}$. But the latter fact follows from Lemma 1.14, parts (i) and (iii).

Assume that k is not a beginning in $\lambda^{(j)}$. Then $y \geq 1$ and $P_j(k) = P_{j+1}(k) + p$. Hence $P_j(l_v) = P_{j+1}(l_v)$ and $P_j(k_v) = P_{j+1}(k_v)+p$ for $v \in [1,y]$. So k_y is a beginning for $\lambda^{(j)}$. The proof of the normalcy of k_y for $\lambda^{(j)}$ is similar to that of the normalcy of k above.

Now assume that k is not good for $\lambda^{(j+1)}$, i.e. there exists a normal bead $m \in (a)_{j+1}$ with $m \succ k$ (in $\lambda^{(j+1)}$). Applying the above arguments to m, one of the following happens:

(1) m is normal in $\lambda^{(j)}$. Then k is not good for $\lambda^{(j)}$ since $m \succ k$ in $\lambda^{(j)}$.

(2) there exists $y' \geq 1$ and beads f_v, e_v such that $P_{j+1}(f_v) = P_{j+1}(m) - 1 + vp$ and $P_{j+1}(e_v) = P_{j+1}(m) + vp$ for $v \in [1, y']$, with $P_{j+1}(m) - 1 + (y'+1)p$ unoccupied in $\lambda^{(j+1)}$ and with $e_{y'}$ normal for $\lambda^{(j)}$. But since none of $e_1, \dots, e_{y'}$ is a beginning in $\lambda^{(j+1)}$, it follows that $k \prec e_{y'}$ in $\lambda^{(j+1)}$, hence in $\lambda^{(j)}$. Thus k is not good in $\lambda^{(j)}$. □

Corollary 3.3. *If there exists a good bead in runner a of $\lambda^{(j+1)}$, then there exists a good bead in runner a of $\lambda^{(j)}$.*

Proof. By the previous lemma if there exists a normal bead in runner a of $\lambda^{(j+1)}$ then there exists a normal bead in runner a of $\lambda^{(j)}$. Now it suffices to recall that acccording to Definition 1.7, any runner which contains a normal bead also contains a good bead. □

Lemma 3.4. *Let $k \in (a)_j$ be a normal bead for $\lambda^{(j)}$ and assume that there exist $u \geq 0$ and beads $e_1, \dots, e_u$ such that $P_j(e_v) = P_j(k) - vp$ for $v \in [1, u]$, $P_j(k) - (u+1)p > 0$, and position $P_j(k) - (u+1)p$ is unoccupied in $\lambda^{(j)}$. Suppose that k is proper for $\lambda^{(j+1)}$ if $u = 0$ and that e_u is proper for $\lambda^{(j+1)}$ if $u > 0$.*

i. *Let k be a beginning for $\lambda^{(j+1)}$. Then k is normal for $\lambda^{(j+1)}$. Moreover, if k is good for $\lambda^{(j)}$ then it is so for $\lambda^{(j+1)}$.*
ii. *Assume that k is not a beginning for $\lambda^{(j+1)}$. Then $u > 0$ and e_u is normal for $\lambda^{(j+1)}$. Moreover, if k is good for $\lambda^{(j)}$ then e_u is so for $\lambda^{(j+1)}$.*

Proof. If $u > 0$ then k is proper for $\lambda^{(j+1)}$ since e_u is so. If $u = 0$ then k is proper for $\lambda^{(j+1)}$ by assumption. Thus k is proper for $\lambda^{(j+1)}$ in any case.

Set

$$L = \{l \in (a-1)_j \mid P_j(l) \geq P_j(k) - 1\}$$

and

$$\begin{aligned} K &= \{k' \in (a)_j \mid k' \prec k\} \\ &= \{k' \in (a)_{j+1} \mid k' \prec k\}. \end{aligned}$$

By Lemma 3.1 we have that position $P_j(k) - 1$ is unoccupied in $\lambda^{(j+1)}$, so k is not a beginning for $\lambda^{(j+1)}$ if and only if $P_{j+1}(k) = P_j(k) - p$, $u > 0$, and position $P_j(k) - p - 1$ is occupied (in $\lambda^{(j+1)}$) by some bead l. So if k is a beginning for $\lambda^{(j+1)}$ then

$$L = \{l \in (a-1)_{j+1} \mid P_{j+1}(l) \geq P_{j+1}(k) - 1\}.$$

By Lemma 1.11, K compensates L in $\lambda^{(j)}$, so K compensates L in $\lambda^{(j+1)}$ in view of 1.16. Now k is normal in $\lambda^{(j+1)}$ by virtue of 1.11.

Next assume that k is not a beginning for $\lambda^{(j+1)}$. By Lemma 3.1 as above, we have $P_{j+1}(e_v) = P_j(e_v) - p$ for $v \in [1,u]$. Since e_u is proper for $\lambda^{(j+1)}$, Lemma 1.12(ii) implies that $P_{j+1}(e_u) - 1$ is unoccupied in $\lambda^{(j+1)}$, i.e. e_u is a beginning for $\lambda^{(j+1)}$; thus $e_u \neq k$ and $u > 0$.

In view of Lemma 1.11, to prove the normalcy of e_u in $\lambda^{(j+1)}$ it suffices to show that

$$K' = \{e_1, \dots, e_{u-1}\} \cup \{k\} \cup K$$

compensates

$$\{l' \in (a-1)_{j+1} \mid P_{j+1}(k) - 1 \geq P_{j+1}(l') \geq P_{j+1}(e_u) - 1\} \cup L.$$

But K compensates L in $\lambda^{(j+1)}$ by Lemma 1.16 and $\{e_1, \dots, e_{u-1}\} \cup \{k\}$ clearly compensates $\{l' \in (a-1)_{j+1} \mid P_{j+1}(k) - 1 \geq P_{j+1}(l') \geq P_{j+1}(e_u) - 1\}$.

The remaining statements of the lemma (those regarding good beads) follow from 3.2. □

4. Degenerate beads

Difficulties arise in the proof of the Main Theorem when the abacus for λ has a degenerate bead. In the next section we show that λ has a degenerate bead if and only if $m(\lambda)$ does; here are some lemmas towards that end.

Lemma 4.1. *Assume that there exists a bead k of λ with $P_0(k) = 1$. Then $\lambda(k)^{(1)} = \lambda^{(1)}$.*

Proof. It follows from the definition that for any bead m with $P_1(m) > P_0(k)$ we have $P_1(m) = P_{\lambda(k)^{(1)}}(m)$. If $P_1(k) = 0$ then $P_1(m) > P_0(k)$ for any bead $m \neq k$. So it suffices to observe that $P_{\lambda(k)^{(1)}}(k) = 0$ since $P_{\lambda(k)}(k) = 0$.

If $P_1(k) = 1$ then there exists a bead f with $P_1(f) = 0$. It is easily seen that $P_1(m) = P_{\lambda(k)^{(1)}}(m)$ for all $m \neq k, f$, $P_{\lambda(k)^{(1)}}(f) = P_1(k)$, and $P_{\lambda(k)^{(1)}}(k) = P_1(f)$, i.e. $\lambda(k)^{(1)} = \lambda^{(1)}$. □

The hypothesis of the next lemma can be depicted schematically as

$$\begin{array}{cccc} & f_x & e_x & \\ \dots & \vdots & \vdots & \dots \\ & f_1 & e_1 & \\ & \cdot & k & \end{array}$$

if $a \not\equiv 0$ and as

$$\begin{array}{ccc} \cdot & \dots & f_x \\ e_x & \dots & f_{x-1} \\ \vdots & \dots & \vdots \\ e_2 & \dots & f_1 \\ e_1 & \dots & \cdot \\ k & \dots & \end{array}$$

if $a \equiv 0$, with f_x in the first row of the abacus in both cases. Notice that if $a \equiv 1$, $P_0(k) > 1$, then the hypotheses of 4.2 cannot hold because λ is a canonical abacus, i.e. position 0 is unoccupied.

Lemma 4.2. *Assume that bead k of λ is r-movable (i.e. $P_1(k) < P_0(k)$) and k is a beginning for λ. Let x be maximal with respect to $P_0(k) - 1 - xp \geq 0$, and assume that there are beads $f_v \in (a-1)_0$, $e_v \in (a)_0$ such that $P_0(f_v) = P_0(k) - 1 - vp$, $P_0(e_v) = P_0(k) - vp$ for $v \in [1, x]$. Then $\lambda(k)^{(1)} = \lambda^{(1)}$.*

Proof. Note that

$$\begin{aligned} P_1(k) &= \min(0, P_0(k) - p) \\ P_1(e_v) &= \min(0, P_0(e_v) - p) \\ P_1(f_v) &= P_0(f_v) \\ P_{\lambda(k)^{(1)}}(k) &= \min(0, P_0(k) - 1 - p) \\ P_{\lambda(k)^{(1)}}(e_v) &= P_0(e_v) \\ P_{\lambda(k)^{(1)}}(f_v) &= \min(0, P_0(f_v) - p) \\ P_{\lambda(k)^{(1)}}(m) &= P_1(m) \end{aligned}$$

for any $v \in [1, x]$ and any bead $m \notin \{k, e_1, \dots, e_x, f_1, \dots, f_x\}$. To prove that $\lambda(k)^{(1)} = \lambda^{(1)}$ it now suffices to observe that $P_{\lambda(k)^{(1)}}(k) = P_1(k) = 0$ if $x = 0$, and

$$\begin{aligned} P_{\lambda(k)^{(1)}}(k) &= P_1(f_1) \\ P_{\lambda(k)^{(1)}}(e_1) &= P_1(k) \\ P_{\lambda(k)^{(1)}}(e_v) &= P_1(e_{v-1}) \text{ for } v \in [2, x] \\ P_{\lambda(k)^{(1)}}(f_v) &= P_1(f_{v+1}) \text{ for } v \in [1, x) \\ P_{\lambda(k)^{(1)}}(f_x) &= P_1(e_x) \end{aligned}$$

if $x > 0$. □

Lemma 4.3. *Assume that $k \in (a)_0$ is a good bead for λ, and μ is the partition such that $\lambda(k)$ is an abacus for μ.*

i. *If k is non-degenerate then there exists a bead $g \in (a)_1$ such that g is good for $\lambda^{(1)}$ and $\lambda(k)^{(1)} = \lambda^{(1)}(g)$.*
ii. *The bead k is degenerate if and only if at least one of the following conditions holds.*
   (a) *$P_0(k) = 1$;*
   (b) *k is r-movable and if x is maximal with respect to $P_0(k) - 1 - xp \geq 0$, then for $v \in [1, x]$, there exist beads f_v, e_v with $P_0(f_v) = P_0(k) - 1 - vp$, $P_0(e_v) = P_0(k) - vp$.*
iii. *μ is p-regular, the first column of the Mullineux symbol $G_p(\mu)$ is $\begin{pmatrix} A_0 \\ R_0 \end{pmatrix}$ if k is non-degenerate, and the first column of the Mullineux symbol $G_p(\mu)$ is*

$\begin{pmatrix} A_0-1 \\ R_0-\xi \end{pmatrix}$ *with* $\xi \in \{0,1\}$ *if* k *is degenerate. Moreover,* $\xi = 1$ *if* $P_0(k) = 1$ *and* $\xi = 0$ *otherwise.*

iv. *Let* k *be degenerate, with* ξ *as in* (iii)*. Then* $A_0 \equiv 0$ *implies* $\xi = 0$*, and* $A_0 \equiv 1$ *implies* $\xi = 1$*.*

Proof. (i) and (ii). Lemmas 4.1 and 4.2 tell us that if either (a) or (b) holds, then k is degenerate. So we assume that neither of (a), (b) holds, and prove that k is non-degenerate and $\lambda(k)^{(1)} = \lambda^{(1)}(g)$ for some bead $g \in (a)_1$ which is good for $\lambda^{(1)}$.

Let y be maximal such that for all $v \in [1,y]$ there exist beads f_v with $P_0(f_v) = P_0(k) - 1 - vp$ (as usual we put $y = 0$ if $P_0(k) - 1 - p$ is unoccupied or $P_0(k) - 1 - p < 0$). Then $y \le x$. Let w be maximal such that there exist beads e_v with $P_0(e_v) = P_0(k) - vp$ for $v \in [1,w]$. Then $w \le x$.

Notice that $w \le y$ since otherwise e_{y+1} would be a normal bead above k, which is impossible since k is good.

Assume first that k is not r-movable. Then k is good for $\lambda^{(1)}$ by Lemma 3.4. Now the fact that $P_0(k) - 1$ is unoccupied in $\lambda^{(1)}$ implies $P_{\lambda(k)^{(1)}}(m) = P_1(m)$ for any bead $m \ne k$, and $P_{\lambda(k)^{(1)}}(k) = P_1(k) - 1$; i.e. $\lambda(k)^{(1)} = \lambda^{(1)}(k)$.

Next assume that k is r-movable. Then $w < x$ since otherwise $w = y = x$ and (b) holds. So $P_1(k) = P_0(k) - p$, and $P_0(e_v) = P_0(e_v) - p$ for $v \in [1,w]$.

If $w = 0$ then k is a beginning for $\lambda^{(1)}$ by 1.12(ii). So it follows from 3.4 that k is good for $\lambda^{(1)}$.

As in the case in which k is not r-movable, we get $P_{\lambda(k)^{(1)}}(m) = P_1(m)$ for any bead $m \ne k$, and $P_{\lambda(k)^{(1)}}(k) = P_1(k) - 1$; i.e. $\lambda(k)^{(1)} = \lambda^{(1)}(k)$.

Let $w > 0$. Then e_w is good for $\lambda^{(1)}$ by 3.4(ii), and we have $\lambda(k)^{(1)} = \lambda^{(1)}(e_w)$.

(iii). Assume that μ is not p-regular. Since λ is p-regular, this means that there are proper beads $m_1, \dots, m_{p-1}$ with $P_0(m_v) = P_0(k) - 1 - v$ for $v \in [1, p-1]$, and $P_0(k) - 1 - p$ is unoccupied. But then m_{p-1} is normal for λ which contradicts the fact that k is good for λ (because $m_{p-1} \succ k$ in λ).

If k is non-degenerate then $P_0(k) > 1$ by (ii). So $\lambda(k)$ has R_0 proper beads. Moreover, $\lambda^{(1)}$ is a partition of $n - A_0$, so $\lambda(k)^{(1)} = \lambda^{(1)}(g)$ implies that $\mu^{(1)}$ is a partition of $n - A_0 - 1$. Notice that μ is a partition of $n-1$. Summarizing, the p-edge of μ has length A_0. The remainder of (iii) is obtained similarly.

(iv) Let $A_0 \equiv 0$. If $P_0(k) = 1$ then by Lemma 3.1, position $P_0(k) - 1$ is unoccupied in $\lambda^{(1)}$. Therefore $R_1 = R_0$. If $\xi = 1$ we get by (iii) that

$$G_p(\mu) = \begin{pmatrix} A_0-1 & A_1 & \dots & A_z \\ R_0-1 & R_1 & \dots & R_z \end{pmatrix}$$

giving a contradiction to 1.1. So $P_0(k) \ne 1$, and $\xi = 0$ by (iii).

Let $A_0 \equiv 1$. If $P_0(k) \ne 1$ then (ii) implies that k is non-degenerate since position 0 is unoccupied in λ. Since k is degenerate, $P_0(k) = 1$; hence $\xi = 1$ by (iii). □

Proposition 4.4. *Let* $(a-1)_1 = \{l_1 \prec \dots \prec l_r\} = L$*. Then runner* a *of* λ *contains a degenerate (good) bead if and only if at least one of the following conditions holds.*

i. $d_0(a) > 0$ *and* L *is compensated in* $\lambda^{(1)}$*;*
ii. $d_0(a) = 2$ *and* $\{l_1, \ldots, l_{r-1}\}$ *is compensated in* $\lambda^{(1)}$.

Proof. Let $K = (a)_1 = \{k_1 \prec \cdots \prec k_s\}$. Assume that runner a of λ contains a degnerate (good) bead k. Then in view of 4.3 at least one of the following conditions holds.

a. $P_0(k) = 1$;
b. k is r-movable and if x is maximal with respect to $P_0(k) - 1 - xp \geq 0$, then for $v \in [1,x]$, there exist beads f_v, e_v with $P_0(f_v) = P_0(k) - 1 - vp$, $P_0(e_v) = P_0(k) - vp$.

Suppose (a) holds. Then $a \equiv 1$. Notice that k is improper for $\lambda^{(1)}$: Otherwise, $R_0 = R_1$, which would imply $p|A_0$ in view of 1.1, and then by 1.3 there exists a bead m_N with $P_1(m_N) = 0$.

Thus $|a|_0 \geq |a|_1 + 1$.

By Lemma 3.1, $A_0 \not\equiv 0$. So in view of Lemmas 2.2, 2.4, and 2.1, we have $|a-1|_0 = |a-1|_1$, i.e. $d_0(a) = |a|_0 - |a|_1 \geq 1$.

If $d_0(a) = 2$ then $P_1(k) = 0$ and for some bead m we have $P_1(m) = 1$, $P_0(k) = P_1(m)$, and $P_0(m) = p+1$, according to Lemma 2.6. From the normalcy of k in λ and Lemma 1.11 we get that $\{m\} \cup K$ compensates L in λ. Hence K compensates $\{l_1, \ldots, l_{r-1}\}$ in λ, and so K compensates $\{l_1, \ldots, l_{r-1}\}$ in $\lambda^{(1)}$ by virtue of 1.16.

If $d_0(a) = 1$ then by 2.7 either $P_1(k) = 0$ and position 1 is unoccupied in $\lambda^{(1)}$, or $P_1(k) = 1$ and there exists a bead m with $P_1(m) = 0$, $P_0(m) \in (0,p)$. In both cases the normalcy of k and Lemmas 1.11, 1.16 imply that K compensates L in $\lambda^{(1)}$.

Now assume that (b) holds.

If $x = 0$ then $|a|_0 = |a|_1 + 1$, $|a-1|_0 = |a-1|_1$, $P_1(k) = 0$, $P_0(k) = \overline{A_0}$ since k is r-movable, and $P_0(k) - 1$ is unoccupied in both λ and $\lambda^{(1)}$. So $d_0(a) = 1$ and

$$L = \{l \in (a-1)_0 \mid P_0(l) \geq P_0(k) - 1\},$$
$$K = \{k' \in (a)_0 \mid k' \prec k\}.$$

By 1.11 the normalcy of k implies that K compensates L in λ, and so K compensates L in $\lambda^{(1)}$ according to Lemma 1.16.

Let $x > 0$ and assume that f_x is proper for $\lambda^{(1)}$. Then e_x is the only new proper bead in $(a)_0$, and runner $a-1$ of λ has no new proper beads. Thus $d_0(a) = 1$ and

$$(a-1)_1 = (a-1)_0 = \{f_1, \ldots, f_x\} \cup L',$$
$$K = \{e_1, \ldots, e_{x-1}\} \cup \{k\} \cup K'$$

where

$$L' = \{l \in (a-1)_1 \mid l \prec f_1\},$$
$$K' = \{k' \in (a)_1 \mid k' \prec k\}.$$

The normalcy of k in λ and Lemma 1.11 imply that K' compensates L' in λ, hence in $\lambda^{(1)}$ in view of 1.16. Also, $\{e_1, \ldots, e_{x-1}\} \cup \{k\}$ clearly compensates $\{f_1, \ldots, f_x\}$ in $\lambda^{(1)}$.

If $x > 0$ and f_x is improper for $\lambda^{(1)}$, we see that f_x is the only new proper bead in runner $a-1$ of λ, and e_x, e_{x-1} (or e_x, k if $x = 1$) are the new proper beads in runner a of λ. So $d_0(a) = 1$. The proof of the fact that $(a)_1$ compensates $(a-1)_1$ in $\lambda^{(1)}$ is similar to the proof in the previous case, using 1.11 and 1.16.

In the other direction, assume that $d_0(a) = 2$ and $\{l_1, \ldots, l_{r-1}\}$ is compensated in $\lambda^{(1)}$. By Lemma 2.6, $a \equiv 1$ and there are beads k, m with $P_1(k) = 0$, $P_1(m) = 1$, $P_0(k) = 1$, and $P_0(m) = p+1$. Let $g_v = k_{s-(r-1-v)}$ for $v \in [1, r-1]$. Then $g_1 \prec \cdots \prec g_{r-1}$ are the top $r-1$ beads in $(a)_1$, and $G = \{g_1, \ldots, g_{r-1}\}$ compensates $L' = \{l_1, \ldots, l_{r-1}\}$ in $\lambda^{(1)}$. If $P_1(g_{r-1}) > 1+p$ then G compensates L' in λ by Lemma 1.14(i). Otherwise, G compensates L' in λ by Lemma 1.14(iii) because $P_1(l_{r-1}) > P_1(l_r) \geq p$. In addition, m compensates l_r in λ since $P_0(l_r) \geq p$. In view of Lemma 1.11, k is normal for λ (hence good); thus (a) holds. By Lemma 4.3 this means that k is degenerate for λ.

Assume $d_0(a) = 1$ and L is compensated in $\lambda^{(1)}$. Let $g_v = k_{s-(r-v)}$, $v \in [1, r]$. Then $G = \{g_1, \ldots, g_r\}$ compensates L in $\lambda^{(1)}$. Since $d_0(a) = 1$ there are some new beads in λ, hence $R_0 - R_1 > 0$ and position 0 is occupied in $\lambda^{(1)}$. Let m be the bead with $P_1(m) = 0$.

In view of Lemma 2.7, one of the following occurs.

- (α) $a \equiv A_0$, $P_0(m) = \overline{A_0}$, and m is the only new proper bead in $(a)_0$;
- (β) $a \equiv A_0 \not\equiv 0, 1$, $P_0(m) = \overline{A_0}$, and there are improper beads l and k with $P_0(m) - 1 = P_1(l)$, $P_0(m) = P_1(k)$, $P_0(l) = P_1(l)$, and $P_0(k) = P_0(m) + p$. In this case m and k are the new proper beads in $(a)_0$ and l is the new proper bead in $(a-1)_0$;
- (γ) $a \equiv 1$, $P_0(m) \in (1, p)$ and there is a bead k with $P_1(k) = 1 = P_0(k)$ which is the only new proper bead in $(a)_0$.

Consider case (α). If $P_0(m) - 1$ is unoccupied in $\lambda^{(1)}$ then K compensates L in λ by Lemma 1.14, parts (i) and (iii), and m is good for λ by Lemma 1.11. So (b) holds, and by Lemma 4.3, k is degenerate for λ.

If $P_1(l_r) = P_0(m) - 1$ then $P_1(g_r) = P_0(m)$ since G compensates L in $\lambda^{(1)}$. Let $u \in [1, r]$ be such that $P_1(l_v) = P_0(l_{v+1}) + p$ for $v \in [u, r)$ and position $P_1(l_u) - p$ is unoccupied in $\lambda^{(1)}$. Then since G compensates L in $\lambda^{(1)}$ we have $P_1(g_v) = P_1(l_v) + 1$ for all $v \in [u, r]$. Then e_u is the top beginning in $(a)_0$. Moreover, e_u is good in view of Lemma 1.14, parts (i) and (iii), and so (b) holds. By Lemma 4.3, e_u is degenerate for λ.

In case (γ) we immediately see that k is good, so (a) holds, and by Lemma 4.3, k is degenerate for λ. Case (β) is similar to (α). ☐

5. Compensation and "steps"

The main result of this section is Corollary 5.8, giving a necessary and sufficient condition for the occurence of a degenerate bead in runner a of λ, in terms of the $d_j(a)$. Throughout the section $0 \leq j \leq z$.

Lemma 5.1. *Let $l \in (a-1)_j$.*

i. *If* $|a|_j = |a|_{j+1} + 1$ *and* $k \in (a)_j$ *is a new bead of* $\lambda^{(j)}$ *then* $\{k\}$ *compensates* $\{l\}$.

ii. *If* $|a|_j = |a|_{j+1} + 2$ *and* $k', k \in (a)_j$, $k' \prec k$ *are new proper beads of* $\lambda^{(j)}$ *then* $\{k\}$ *compensates* $\{l\}$ *and* $\{k, k'\}$ *compensates any subset* $L \subseteq (a-1)_j$ *with* $|L| = 2$.

Proof. It follows from Lemmas 2.2 and 2.4 that $P_j(l) \geq P_j(k) - 1$ for any $l \in (a)_j$, giving (i). To prove (ii) we have also to notice that $P_j(k') = P_j(k) + p$ in view of Lemma 2.2. □

Lemma 5.2. *Assume that* $d_j(a) < 0$, *and let* L *be a subset of* $(a-1)_j$ *with* $|L| \geq |a-1|_j + d_j(a) + 1$. *Then* L *is not compensated in* $\lambda^{(j)}$.

Proof. Let $(a-1)_j = \{l_1 \prec \cdots \prec l_r\}$.

Assume that $d_j(a) = -2$. We must show that if $|L| \geq r - 1$, then L is not compensated in $\lambda^{(j)}$. It is clearly sufficient to prove that

$$L' = \{l_1 \prec \cdots \prec l_{r-1}\}$$

is not compensated in $\lambda^{(j)}$.

It follows from Lemma 2.9 that l_{r-1} and l_r are the new proper beads of $(a-1)_j$, and that $(a)_j$ does not have new proper beads, i.e. $(a)_j = (a)_{j+1}$. Moreover, by the same lemma, l_{r-1} is an improper bead in runner $a-1$ of $\lambda^{(j+1)}$, and

$$P_{j+1}(l_r) < P_j(l_r) = P_{j+1}(l_{r-1}) = P_j(l_{r-1}) - p.$$

Notice that position $P_{j+1}(l_{r-1}) + 1$ is unoccupied in $\lambda^{(j+1)}$, since if $P_{j+1}(k) = P_{j+1}(l_{r-1}) + 1$ then k would be a new proper bead in $(a)_j$. Therefore $P_{j+1}(l_{r-1}) + 1$ is also unoccupied in $\lambda^{(j)}$.

Let w be the smallest integer such that

$$P_{j+1}(l_v) = P_{j+1}(l_{v+1}) + p \text{ for } v \in [w, r-1),\ P_{j+1}(l_w) + p \ \text{ is unoccupied in } \lambda^{(j)}.$$

Then $P_j(l_v) = P_{j+1}(l_v) + p,\ v \in [w, r-1]$ and $P_j(l_w) + 1$ is unoccupied in $\lambda^{(j)}$ by Lemma 1.12(i). Now it follows that L' is not compensated in $\lambda^{(j)}$ since

$$\begin{aligned} &|\{l \in L' \mid P_j(l) \leq P_j(l_w)\}| = |\{l_w, \ldots, l_{r-1}\}| = r - w, \\ &|\{k \in (a)_j \mid P_j(k) \leq P_j(l_w) + 1\}| \leq r - w - 1. \end{aligned}$$

If $d_j(a) = -1$ we have to prove that $\{l_1 \prec \cdots \prec l_r\}$ is not compensated in $\lambda^{(j)}$. This is similar to the case $d_j(a) = -2$ above, using Lemma 2.8 instead of Lemma 2.9. □

Notation. For the remainder of the section we assume that

$$(a-1)_0 = \{l_1 \prec \cdots \prec l_s\}$$

and put $j_v = \mathrm{St}(l_v)$ for $v \in [1, s]$. Then $j_1 \geq \cdots \geq j_s$. (cf. Definition 1.17 and the remark following it.)

Definition 5.3. We define an integer $M \in [0,s]$, integers $h_1, \dots, h_M$, and beads $k_1 \prec \cdots \prec k_M \in (a)_0$ inductively as follows.

If $\mathrm{St}(k) > j_1$ for all $k \in (a)_0$ we set $M = 0$. Otherwise let k_1 be the lowest bead in $(a)_0$ such that $\mathrm{St}(k_1) \le j_1$.

Assume we have already defined $k_1, \dots, k_t$. If either (1) $t = s$ or (2) $t < s$ but $\mathrm{St}(k) > j_{t+1}$ for all $k \in (a)_0$ with $k \succ k_t$, then we put $M = t$. Otherwise, let k_{t+1} be the lowest bead in $(a)_0$ such that $\mathrm{St}(k_{t+1}) \le j_{t+1}$ and $k_{t+1} \succ k_t$.

Finally put $h_v = \mathrm{St}(k_v)$ for $v \in [1,M]$.

Proposition 5.4. *Assume $r \le s$ and $j \le j_r$ (i.e. $l_r \in (a-1)_j$). Then $L = \{l_1, \dots, l_r\}$ is compensated in $\lambda^{(j)}$ if and only if $M \ge r$ and $h_r \ge j$ (i.e. $k_r \in (a)_j$).*

Proof. We proceed by induction on r. Let $r = 1$.

Assume $\{l_1\}$ is compensated in $\lambda^{(j)}$. We may suppose that either (1) $j = j_1$ or (2) $j < j_1$ but $\{l_1\}$ is not compensated in $\lambda^{(j+1)}$. We must show that $M \ge 1$ and $h_1 \ge j$. Assume not.

We claim that $\mathrm{St}(k) > j$ for any $k \in (a)_j$. Indeed, if $M = 0$ then by Definition 5.3, $\mathrm{St}(k) > j_1 \ge j$ for any $k \in (a)_0 \supseteq (a)_j$. If $M = 1$ and $h_1 < j$, then $k_1 \notin (a)_j$. But $\mathrm{St}(k) = j$ for some $k \in (a)_j$ implies $k \prec k_1$ in λ and $\mathrm{St}(k) = j \le j_1$, which contradicts Definition 5.3.

Thus $|a|_j = |a|_{j+1}$. Assume that $j = j_1$. Then $|a-1|_{j+1} = 0$ (since l_1 is the lowest proper bead in runner a), and $|a-1|_j > 0$. Thus $d_j(a) = -|a-1|_j < 0$ and $\{l_1\}$ is not compensated in $\lambda^{(j)}$ by Lemma 5.2, contrary to our assumption.

Now assume that $j < j_1$ and $\{l_1\}$ is not compensated in $\lambda^{(j+1)}$. Let $k \in (a)_j$ be a bead such that $\{k\}$ compensates $\{l_1\}$ in $\lambda^{(j)}$. If $k \in (a)_{j+1}$ then $\{k\}$ would compensate $\{l_1\}$ in $\lambda^{(j+1)}$ by Lemma 1.16. Hence $\mathrm{St}(k) = j$. Therefore $M \ge 1$ and $h_1 \ge j$ since $k_1 \preceq k$ by definition.

In the other direction, let $M \ge 1$, $h_1 \ge j$. Since $j_1 \ge h_1$ then $l_1 \in (a-1)_{h_1}$. Moreover, by Lemma 5.1, $\{l_1\}$ is compensated in $\lambda^{(h_1)}$. By Corollary 1.15, $\{l_1\}$ is compensated in $\lambda^{(j)}$.

Now for the inductive step: Let $r > 1$ and assume that the proposition has been proved for all subsets $\{l_1, \dots, l_{r'}\}$ of $(a-1)_{j'}$ for all j' and all $r' < r$.

Assume that L is compensated in $\lambda^{(j)}$. We may suppose that either $j_r = j$ (i.e. $L \not\subseteq (a-1)_{j+1}$) or $j_r > j$ but L is not compensated in $\lambda^{(j+1)}$. Since L is compensated in $\lambda^{(j)}$ then it is compensated by the set of the top r beads, say $m_1 \prec \cdots \prec m_r$, of $(a)_j$.

By induction we have $M \ge r-1$, $h_{r-1} > j$. Assume that $M = r-1$, or $M \ge r$ but $h_r < j$.

Note that either $m_r = k_{r-1}$ or $\mathrm{St}(m_r) > j_r$ (or both) since otherwise we would have $M \ge r$, $h_r \ge j$ by Definition 5.3.

If $\mathrm{St}(m_r) > j$ then $\{m_1, \dots, m_r\} \subseteq (a)_{j+1}$. So in view of Lemma 1.16, $L \subseteq (a-1)_{j+1}$ and $\{m_1 \dots, m_r\}$ compensates L in $\lambda^{(j+1)}$. Thus $\mathrm{St}(m_r) = j$ and $m_r = k_{r-1}$.

We consider three cases.

(a) $j_{r-1} > j$ and $\{l_1, \dots, l_{r-1}\}$ is compensated in $\lambda^{(j+1)}$. By the inductive hypothesis, $h_{r-1} \geq j+1$. But $h_{r-1} = \mathrm{St}(k_{r-1}) = \mathrm{St}(m_r) = j$, giving a contradiction.

(b) $j_{r-1} > j$ and $\{l_1, \dots, l_{r-1}\}$ is not compensated in $\lambda^{(j+1)}$. Now if $\mathrm{St}(m_{r-1}) > j$ then $\{m_1, \dots, m_{r-1}\} \subseteq (a)_{j+1}$ and so $\{m_1, \dots, m_{r-1}\}$ compensates $\{l_1, \dots, l_{r-1}\}$ by Lemma 1.16. So $\mathrm{St}(m_{r-1}) = j$. We claim that $r \geq 3$ and $m_{r-1} = k_{r-2}$. Indeed, if $r = 2$ we have $\mathrm{St}(m_1) = \mathrm{St}(m_2) = j$, $\mathrm{St}(l_1) \geq j$, and $\mathrm{St}(l_2) \geq j$. Hence $k_1 \preceq m_1 \prec m_2 = k_1$, giving a contradiction. If $r \geq 3$ and $k_{r-2} \prec m_{r-1}$, then $\mathrm{St}(m_{r-1}) = \mathrm{St}(m_r) = j$ implies $k_{r-1} \preceq m_{r-1}$, giving a contradiction since $k_{r-1} = m_r$. Thus $k_{r-2} = m_{r-1}$. Therefore $h_{r-2} = j < j+1$. By the inductive hypothesis, $\{l_1, \dots, l_{r-2}\}$ is not compensated in $\lambda^{(j+1)}$. But $\{m_1, \dots, m_{r-2}\} \subseteq (a)_{j+1}$ and so $\{m_1, \dots, m_{r-2}\}$ compensates $\{l_1, \dots, l_{r-2}\}$ in $\lambda^{(j+1)}$ by Lemma 1.16.

(c) $j_{r-1} = j$. Then also $j_r = j$. Thus we get $|a-1|_j = |a-1|_{j+1} + 2$. Hence $d_j(a) < 0$ because at most one runner can grow by two beads with the addition of a p-edge, cf. Corollary 2.3. By Lemma 5.2, L is not compensated in $\lambda^{(j)}$.

In the other direction, assume that $M \geq r$ and $h_r \geq j$. By Corollary 1.15 we may suppose that $j = h_r$ (taking into account that $j_r \geq h_r$).

We have $j_{r-1} \geq j_r \geq j$. If $j_{r-1} = j$ then $j_r = j$; and $h_{r-1} \geq h_r = j$, $h_{r-1} \leq j_{r-1}$ by definition. So $h_{r-1} = h_r = j$ which contradicts Corollary 2.3. So $j_{r-1} > j$, i.e. $\{l_1, \dots, l_{r-1}\} \subseteq (a-1)_{j+1}$.

We consider two cases.

(a) $h_{r-1} = j$. If $r = 2$ then by Lemma 5.1(ii), $\{k_1, k_2\}$ compensates L in $\lambda^{(j)}$. Let $r > 2$. Since $|a|_j \leq |a|_{j+1} + 2$ by 2.1, we have $h_{r-2} > j$. By the inductive hypothesis $L' = \{l_1, \dots, l_{r-2}\}$ is compensated in $\lambda^{(j+1)}$. Let $N = \{n_1 \prec \cdots \prec n_{r-2}\}$ be the set of the top $r-2$ beads in $(a)_{j+1}$. Then N compensates L' in $\lambda^{(j+1)}$. We noted above that l_{r-1} is a proper bead in runner $a-1$ of $\lambda^{(j+1)}$. So either $P_{j+1}(n_{r-2}) < P_{j+1}(l_{r-2}) + 1$, or position $P_{j+1}(n_{r-2}) - p$ is unoccupied in $\lambda^{(j+1)}$ (since n_{r-2} is the top (proper) bead of $(a)_{j+1}$). Now N compensates L' in $\lambda^{(j)}$ by Lemma 1.14(iii) in the first case and by Lemma 1.14(i) in the second. Now it suffices to notice that $\{k_{r-1}, k_r\}$ compensates $\{l_{r-1}, l_r\}$ in $\lambda^{(j)}$ by Lemma 5.1, and $\{k_{r-1}, k_r\} \cap N = \emptyset$ since $\mathrm{St}(k_{r-1}) = \mathrm{St}(k_r) = j$.

(b) $h_{r-1} > j$. Then by the inductive hypothesis $L' = \{l_1, \dots, l_{r-1}\}$ is compensated in $\lambda^{(j+1)}$. Let $N = \{n_1 \prec \cdots \prec n_{r-1}\}$ be the set of the top $r-1$ beads in $(a)_{j+1}$. If $P_{j+1}(l_{r-1}) > P_{j+1}(n_{r-1}) - 1$ or there does not exist a bead n with $P_j(n) = P_{j+1}(n_{r-1})$ then N compensates L' in $\lambda^{(j)}$ by Lemma 1.14, parts (i) and (iii), and the top new bead in $(a)_j$ compensates $\{l_r\}$ in $\lambda^{(j)}$ by Lemma 5.1. So we may assume that $P_{j+1}(l_{r-1}) = P_{j+1}(n_{r-1}) - 1$ and $P_j(n) = P_{j+1}(n_{r-1})$ for some necessarily improper bead n of $\lambda^{(j+1)}$. Note that $l_r \notin (a-1)_{j+1}$ since otherwise $P_{j+1}(n_{r-1}) - p$ would be unoccupied in $\lambda^{(j+1)}$, giving a contradiction. For the same reason we have that either $P_{j+1}(l_{r-1}) < p$ or $P_{j+1}(l_{r-1}) - p$ is occupied by an improper bead.

Notice that $P_j(l_{r-1}) = P_{j+1}(l_{r-1})$ because otherwise n_{r-1} would not be a-movable in $\lambda^{(j+1)}$. Since $l_r \succ l_{r-1}$, $l_r \in (a-1)_j$ we get $P_j(l_{r-1}) > p$. So position $P_{j+1}(l_{r-1}) - p$ is occupied by the improper bead l_r in $\lambda^{(j+1)}$ (l_r cannot be in a runner of $\lambda^{(j+1)}$ other than a because that would force $P_j(l_r) = P_{j+1}(l_{r-1})$, contradicting $P_j(l_{r-1}) = P_{j+1}(l_{r-1})$); that is, $P_{j+1}(l_r) = P_{j+1}(l_{r-1}) - p$. Since $P_j(l_{r-1}) = P_{j+1}(l_{r-1})$, we have also $P_j(l_r) = P_{j+1}(l_r)$.

We claim that $P_{j+1}(n) = P_{j+1}(n_{r-1}) - p$. Indeed, $P_{j+1}(n) \geq P_j(n) - p = P_{j+1}(n_{r-1}) - p$, and if n belongs to a runner different from a in $\lambda^{(j+1)}$ then l_r is not proper for $\lambda^{(j)}$.

Since l_r is proper for $\lambda^{(j)}$ we also conclude that there exists some bead m with $P_{j+1}(m) < P_{j+1}(l_r)$, and $P_j(m) = P_{j+1}(n)$ (then $m \in (a)_j$ since $P_j(m) > P_j(l_r)$).

Now $\{n, n_2, \ldots, n_{r-1}\}$ compensates L' in $\lambda^{(j)}$ by Lemma 1.14(ii) and $\{m\}$ compensates $\{l_r\}$ in $\lambda^{(j)}$ by Lemma 5.1. ☐

Recall that z was introduced in Section 1 as the length of the Mullineux symbol for λ, and the integers $j_v = \mathrm{St}(l_v)$ were defined before 5.3.

Lemma 5.5. *Let $(a-1)_j = \{l_1 \prec \cdots \prec l_r\}$ and $q \leq r$. The subset $L = \{l_1, \ldots, l_q\} \subseteq (a-1)_j$ is compensated in $\lambda^{(j)}$ if and only if for every $h \in [j_q, z]$,*

$$|\{l \in L \mid \mathrm{St}(l) \leq h\}| \leq |\{k \in (a)_j \mid \mathrm{St}(k) \leq h\}| \tag{2}$$

Proof. Assume that L is compensated in $\lambda^{(j)}$. Then by Proposition 5.4, $M \geq q$ and $h_q \geq j$ (hence $k_q \in (a)_j$). Let $h \in [j_q, z]$. If w is minimal with respect to $\mathrm{St}(l_w) \leq h$, then the set on the left hand side of (2) is $\{l_w, \ldots, l_q\}$. However the set on the right hand side of (2) contains $\{k_w, \ldots, k_q\}$ since $\mathrm{St}(k_v) \leq \mathrm{St}(l_v)$ by definition.

Conversely, assume that (2) holds for any $h \in [j_q, z]$. We want to show that $M \geq q$ and $h_q \geq j$, and then to apply Proposition 5.4.

We may assume $q \geq 1$. By induction on $t = 1, 2, \ldots, q$, we show that $M \geq t$, $h_t \geq j$: Put $h = j_1 \in [j_q, z]$. It follows from (2) that $M \geq 1$, $h_1 \geq j$. Suppose $t \leq q$ and we have already proved that $M \geq t-1$, $h_{t-1} \geq j$. Put $h = j_t \in [j_q, z]$. Then it follows from (2) that $M \geq t$ and $h_t \geq j$ (cf. Definition 5.3). ☐

Notation. Let $(a-1)_j = \{l_1 \prec \cdots \prec l_r\}$, $q \leq r$ and $h \in [j_q, z]$. We denote

$$L(h) = L_j(q, h) = \{l \in \{l_1, \ldots, l_q\} \mid \mathrm{St}(l) \leq h\};$$

$$K(h) = K_j(h) = \{k \in (a)_j \mid \mathrm{St}(k) \leq h\}.$$

Lemma 5.6. $|K(h)| - |L(h)| = (r-q) + \sum_{i=j}^{h} d_i(a)$.

Proof. Note that

$$|K(h)| = |a|_j - |a|_{h+1} \text{ and}$$
$$|L(h)| = |a-1|_j - |a-1|_{h+1} - (r-q).$$

So

$$|K(h)| - |L(h)| = |a|_j - |a|_{h+1} - (|a-1|_j - |a-1|_{h+1}) + (r-q).$$

To complete the proof of the lemma it suffices to recall that $d_i(a) = |a|_i - |a|_{i+1} - (|a-1|_i - |a-1|_{i+1})$. □

Corollary 5.7. *Let $(a-1)_j = \{l_1 \prec \cdots \prec l_r\}$, $q \leq r$. The subset $L = \{l_1, \dots, l_q\} \subseteq (a-1)_j$ is compensated in $\lambda^{(j)}$ if and only if for every $h \in [j, z]$,*

$$(r-q) + \sum_{i=j}^{h} d_i(a) \geq 0.$$

Proof. In view of Lemmas 5.5 and 5.6 we have only to prove that if L is compensated in $\lambda^{(j)}$, then $(r-q) + \sum_{i=j}^{h} d_i(a) \geq 0$ for every $h \in [j, j_q)$. For such h write

$$\begin{aligned}(r-q) + \sum_{i=j}^{h} d_i(a) &= (r-q) + |a|_j - |a|_{h+1} - (|a-1|_j - |a-1|_{h+1}) \\ &\geq (r-q) - (|a-1|_j - |a-1|_{h+1}).\end{aligned}$$

But

$$\begin{aligned}|a-1|_j - |a-1|_{h+1} &= |\{l \in \{l_1, \dots, l_r\} \mid \mathrm{St}(l) \in [j, h]\}| \\ &\leq |\{l \in \{l_1, \dots, l_r\} \mid \mathrm{St}(l) \in [j, j_q)\}| \\ &\leq |\{l_{q+1}, \dots, l_r\}| = r - q.\end{aligned}$$

□

Corollary 5.8. *Runner a of λ contains a degenerate good bead if and only if*

$$\sum_{i=0}^{h} d_i(a) > 0 \quad \textit{for any } h \in [0, z].$$

Proof. This follows immediately from Corollary 5.7 and Proposition 4.4 □

6. Neighbour runner sizes for λ and $m(\lambda)$.

The purpose of this section is to prove Corollary 6.7, which expresses $|A_0 + \varepsilon_0 - a|^j - |A_0 + \varepsilon_0 - a - 1|^j$ in terms of the $|a|_j - |a-1|_j$. (Recall that we use superscripts to represent quantities associated with $m(\lambda)$, and subscripts for λ.)

Let γ be a p-regular partition and let Γ be an abacus for γ. Throughout this section we use elements of the residue system $[0, p)$ to represent residue classes modulo p, when discussing runner numbers.

Notation. Denote by $\widetilde{\Gamma}$ the abacus obtained from Γ by moving all beads up along their runners as far as they can go.

Example. If

$$\Gamma = \begin{array}{ccccc} \cdot & k & \cdot & \cdot & l \\ \cdot & m & \cdot & n & \cdot \\ q & \cdot & r & s & \cdot \\ \cdot & t & u & \cdot & v \end{array}$$

then

$$\widetilde{\Gamma} = \begin{matrix} q & k & r & n & l \\ \cdot & m & u & s & v \\ \cdot & t & & & \end{matrix}$$

We need to recall some notions concerning p-cores of Young diagrams. Details can be found in [12].

A rim p-hook of a Young diagram is a connected part of its rim, having length p, and such that after its removal, what remains is again a Young diagram. A partition is called a p-core if and only if its Young diagram does not have any rim p-hooks. For any partition γ one can remove rim p-hooks from its Young diagram one after another (in any order) until a diagram of a p-core is obtained. The latter partition (which is well defined, cf. [12]) is called the p-core of γ. We denote it by $\mathrm{core}(\gamma)$.

We make use of the following results.

Proposition 6.1. [10] *Let Γ be an abacus for γ.*

i. *There is a 1-1 correspondence between rim p-hooks of γ and beads m of Γ such that the position immediately above m (i.e. position $P_\Gamma(m) - p$) is unoccupied in Γ. Moreover, the abacus obtained from Γ by moving such a bead m up one position is an abacus for the partition obtained from γ by removing the corresponding rim p-hook.*

ii. *γ is a p-core if and only if whenever position f of Γ is occupied, all positions of the form $f - vp$ are occupied (i.e. $\Gamma = \widetilde{\Gamma}$).*

iii. *$\widetilde{\Gamma}$ is an abacus for* $\mathrm{core}(\gamma)$.

Proposition 6.2. [22] *If γ is a p-regular partition then* $\mathrm{core}(m(\lambda)) = \mathrm{core}(\lambda)'$ *(recall that "$\,'\,$" means conjugation).*

Notation. i. For any abacus Γ we denote by $\{a\}_\Gamma$ the number of *all* (not just proper) beads in runner a of Γ.

ii. Let Γ be an abacus for a p-core, cf. Proposition 6.1(ii), and let

$$d = \max_{a \in [0,p)} \{a\}_\Gamma.$$

We define an abacus Γ' via the following:

(1) Γ' is an abacus for a p-core;

(2) $\{a\}_{\Gamma'} = d - \{p-1-a\}_\Gamma$ for all $a \in [0,p)$.

Lemma 6.3. *Let Γ be an abacus for a p-core partition γ. Then Γ' is an abacus for γ'.*

Proof. Let Γ have T beads. Then Γ is constructed from the sequence $s_T(\gamma) = \{s_0, s_1, \ldots, s_v, \ldots\}$ of spaces and beads as explained in Section 1.

If U is such that $s_v =$ "$\cdot$" for any $v > U$, we define a new sequence $s_{T,U}(\gamma) = \{t_0, t_1, \ldots, t_v, \ldots\}$ by setting

$$t_v = \begin{cases} \text{“}\cdot\text{”} & \text{if } v > U \text{ or } v \le U,\ s_{U-v} = \text{“}\times\text{”} \\ \text{“}\times\text{”} & \text{otherwise.} \end{cases}$$

Example. If $\gamma = (4, 2^2, 1^3)$, $T = 7$, $U = 15$ then

$$s_T(\gamma) = \times \quad \cdot \quad \times \quad \times \quad \times \quad \cdot \quad \times \quad \times \quad \cdot \quad \cdot \quad \times,$$
$$s_{T,U}(\gamma) = \times \quad \times \quad \times \quad \times \quad \cdot \quad \times \quad \times \quad \cdot \quad \cdot \quad \times \quad \cdot \quad \cdot \quad \cdot \quad \times.$$

It follows from the definitions that $s_{T,U}(\gamma) = s_{U-T}(\gamma')$. Now it remains to notice that if U is minimal with respect to $s_v =$ "$\cdot$" for any $v > U$ and $U \equiv 0$, then the abacus constructed from $s_{T,U}(\gamma)$ is Γ'. □

Lemma 6.4. *Let Γ be an abacus with T beads, V of which are proper. Then*

$$|a|_\Gamma - |a-1|_\Gamma = \begin{cases} \{a\}_\Gamma - \{a-1\}_\Gamma - 1, & \text{if} \quad T - V \not\equiv 0,\ a \equiv 0; \\ \{a\}_\Gamma - \{a-1\}_\Gamma + 1, & \text{if} \quad T - V \not\equiv 0,\ a \equiv T - V; \\ \{a\}_\Gamma - \{a-1\}_\Gamma, & \text{otherwise.} \end{cases}$$

Proof. The improper beads of Γ occupy positions from 0 to $T - V - 1$ in Γ. So if we write $T - V = Qp + R$ with $Q, R \in \mathbb{Z}$, $0 \le R < p$, then

$$|a|_\Gamma = \begin{cases} \{a\}_\Gamma - Q, & \text{if} \quad a \equiv x \text{ with } 0 \le x < p,\ x \ge R; \\ \{a\}_\Gamma - Q - 1, & \text{if} \quad a \equiv x \text{ with } 0 \le x < p,\ x < R. \end{cases}$$

The lemma follows. □

Lemma 6.5. *Let Γ_1, Γ_2 be two abaci for γ with T_1 and T_2 beads, respectively. Then $|a|_{\Gamma_1} = |a + T_2 - T_1|_{\Gamma_2}$ for any $a \in [0, p)$.*

Proof. Assume without loss of generality that $T_2 \ge T_1$. Let $s_{T_1}(\gamma) = \{s_0, s_1, \dots, s_v, \dots\}$, $s_{T_2}(\gamma) = \{t_0, t_1, \dots, t_v, \dots\}$. Then $t_v =$ "$\times$" for $v \in [0, T_2 - T_1)$ and $t_{v+T_2-T_1} = s_v$ for $v \ge 0$. The lemma follows. □

Proposition 6.6. *Let Γ be the abacus with R beads for a p-regular partition γ and let Ω be the abacus with S beads for $m(\gamma)$. Assume that r beads of Γ are proper and s beads of Ω are proper. For any $a \in [0, p)$, put $\hat{a} \equiv R + S - a$. Then*

$$|\hat{a}|_\Omega - |\hat{a}-1|_\Omega = \begin{cases} |a|_\Gamma - |a-1|_\Gamma - 1, & \text{if} \quad a \equiv R - r,\ r + s \not\equiv 0; \\ |a|_\Gamma - |a-1|_\Gamma + 1, & \text{if} \quad a \equiv R + s,\ r + s \not\equiv 0; \\ |a|_\Gamma - |a-1|_\Gamma, & \text{otherwise.} \end{cases}$$

Proof. For any abacus Λ we let $\delta_\Lambda|a| = |a|_\Lambda - |a-1|_\Lambda$ and $\delta_\Lambda\{a\} = \{a\}_\Lambda - \{a - 1\}_\Lambda$.

Note that it suffices to prove that $\delta_\Omega|\hat{a}| - \delta_\Gamma|a| \in \{0, \pm 1\}$ and that $\delta_\Omega|\hat{a}| = \delta_\Gamma|a| + 1$ is equivalent to $a \equiv R + s$, $r + s \not\equiv 0$. Indeed, if this is proved then applying it to Γ and Ω interchanged we have that $\delta_\Omega|\hat{a}| = \delta_\Gamma|a| - 1$ is equivalent to $\hat{a} \equiv S + r$, $r + s \not\equiv 0$, which is in turn equivalent to $a \equiv R - r$, $r + s \not\equiv 0$ (since $\hat{a} \equiv R + S - a$).

By definition we have

$$\{a\}_\Gamma = \{a\}_{\tilde{\Gamma}} = d - \{p - 1 - a\}_{\tilde{\Gamma}'}.$$

Hence

$$\delta_{\tilde{\Gamma}'}\{-a\} = \delta_\Gamma\{a\}. \tag{3}$$

By Proposition 6.1(iii), $\widetilde{\Gamma}$ is an abacus for $\mathrm{core}(\gamma)$. So Proposition 6.2 and Lemma 6.3 imply that $\widetilde{\Gamma}'$ is an abacus for $\mathrm{core}(m(\gamma))$. Let $\widetilde{\Gamma}'$ have U beads. Then $U \equiv -R$ since $\widetilde{\Gamma}$ has R beads.

Let Ω_1 be an abacus for $m(\gamma)$ having $V \equiv -R$ beads. Then by Proposition 6.1(iii), $\widetilde{\Omega}_1$ is an abacus for $\mathrm{core}(m(\gamma))$ having V beads. Since $V \equiv U$, we have, for any $b \in [0,p)$:

$$\delta_{\widetilde{\Omega}_1}\{b\} = \delta_{\widetilde{\Gamma}'}\{b\}. \tag{4}$$

Since $\{b\}_{\Omega_1} = \{b\}_{\widetilde{\Omega}_1}$, equations (3) and (4) give us

$$\delta_{\Omega_1}\{-a\} = \delta_{\Gamma}\{a\}. \tag{5}$$

Now we use Lemma 6.4. We have

$$\delta_{\Omega_1}|-a| = \begin{cases} \delta_{\Omega_1}\{-a\} - 1, & \text{if } -R-s \not\equiv 0,\ -a \equiv 0; \\ \delta_{\Omega_1}\{-a\} + 1, & \text{if } -R-s \not\equiv 0,\ -a \equiv -R-s; \\ \delta_{\Omega_1}\{-a\}, & \text{otherwise}; \end{cases} \tag{6}$$

and

$$\delta_{\Gamma}\{a\} = \begin{cases} \delta_{\Gamma}|a| + 1, & \text{if } R-r \not\equiv 0,\ a \equiv 0; \\ \delta_{\Gamma}|a| - 1, & \text{if } R-r \not\equiv 0,\ a \equiv R-r; \\ \delta_{\Gamma}|a|, & \text{otherwise}. \end{cases} \tag{7}$$

It follows from (5)–(7) that $\delta_{\Omega_1}|-a| - \delta_{\Gamma}|a| \in \{0, \pm 1, \pm 2\}$. Moreover, $\delta_{\Omega_1}|-a| = \delta_{\Gamma}|a| + 2$ if and only if $-R-s \not\equiv 0,\ -a \equiv -R-s$ and $R-r \not\equiv 0,\ a \equiv 0$, which is impossible. Similarly $\delta_{\Omega_1}|-a| = \delta_{\Gamma}|a| - 2$ is impossible.

Let us prove that $\delta_{\Omega_1}|-a| = \delta_{\Gamma}|a| + 1$ if and only if $a \equiv R+s,\ r+s \not\equiv 0$. Then the proposition will follow since $\delta_{\Omega}|-a+S-V| = \delta_{\Omega_1}|-a|$ in view of Lemma 6.5, and $-a+S-V \equiv -a+S+R \equiv \hat{a}$.

It follows from (5)–(7) that $\delta_{\Omega_1}|-a| = \delta_{\Gamma}|a| + 1$ if and only if one of the following conditions holds.

(1) $-R-s \not\equiv 0,\ -a \equiv -R-s,\ R-r \equiv 0$;
(2) $-R-s \not\equiv 0,\ -a \equiv -R-s,\ R-r \not\equiv 0,\ a \not\equiv 0,\ a \not\equiv R-r$;
(3) $-R-s \equiv 0,\ R-r \not\equiv 0,\ a \equiv 0$;
(4) $-R-s \not\equiv 0,\ -a \not\equiv 0,\ -a \not\equiv -R-s,\ R-r \not\equiv 0,\ a \equiv 0$.

Now it suffices to remark that
(1) is equivalent to $a \equiv R+s,\ r+s \not\equiv 0,\ a \equiv r+s$;
(2) is equivalent to $a \equiv R+s,\ r+s \not\equiv 0,\ a \not\equiv 0,\ a \not\equiv r+s$;
(3) is equivalent to $a \equiv R+s,\ r+s \not\equiv 0,\ a \equiv 0$;
(4) is impossible. □

Corollary 6.7. *Put $\hat{a} = A_0 + \varepsilon_0 - a$. Then*

$$|\hat{a}|^j - |\hat{a}-1|^j = \begin{cases} |a|_j - |a-1|_j - 1, & \text{if} \quad a \equiv R_0 - R_j,\ A_j \not\equiv 0,\ A_j \not\equiv -1; \\ |a|_j - |a-1|_j + 1, & \text{if} \quad a \equiv R_0 + S_j,\ A_j \not\equiv 0,\ A_j \not\equiv -1; \\ |a|_j - |a-1|_j, & \text{otherwise}. \end{cases}$$

Proof. Recall that $R_j+S_j=A_j+\varepsilon_j$. Moreover, $A_j+\varepsilon_j\not\equiv 0$ if and only if $A_j\not\equiv 0$, $A_j\not\equiv -1$. Now it suffices to recall that $\lambda^{(j)}$ (resp. $m(\lambda)^{(j)}$) is the abacus with R_0 (resp. S_0) beads, R_j (resp. S_j) of which are proper, and apply Proposition 6.6. □

7. The occurrence of degenerate beads for λ and $m(\lambda)$.

The main result of this section says that runner a of λ has a degenerate good bead if and only if runner $\hat{a}$ of $m(\lambda)$ does. Throughout, $j\in[0,z]$ and $\hat{a}\equiv A_0+\varepsilon_0-a$.

Lemma 7.1. *Let $d_j(a)>0$. Then $d^j(\hat{a})\geq d_j(a)-1$, and if $d^j(\hat{a})=d_j(a)-1$ then*

$$j<z,\ a\equiv R_0+S_{j+1},\ A_{j+1}\not\equiv 0,\ \text{and } A_{j+1}\not\equiv -1.$$

Proof. If $a\equiv R_0-R_j$ and $A_j\not\equiv 0$, then it follows from Lemmas 2.5, 2.6, and 2.7 that $d_j(a)\leq 0$, giving a contradiction. So Corollary 6.7 implies

$$|\hat{a}|^j-|\hat{a}-1|^j=|a|_j-|a-1|_j+x$$

where $x=1$ if $a\equiv R_0+S_j$, $A_j\not\equiv 0$, $A_j\not\equiv -1$, and $x=0$ otherwise. Again by 6.7 we have

$$|\hat{a}|^{j+1}-|\hat{a}-1|^{j+1}-(|a|_{j+1}-|a-1|_{j+1})\in\{0,\pm 1\}.$$

Summarizing, we have $d^j(\hat{a})\geq d_j(a)-1$; and $d^j(\hat{a})=d_j(a)-1$ implies

$$|\hat{a}|^{j+1}-|\hat{a}-1|^{j+1}=|a|_{j+1}-|a-1|_{j+1}+1.$$

Therefore $j+1\leq z$ and, in view of 6.7, $a\equiv R_0+S_{j+1}$, $A_{j+1}\not\equiv 0$, and $A_{j+1}\not\equiv -1$. □

Lemma 7.2. *Let $d^j(\hat{a})<0$. Then $d_j(a)\leq 0$, and if $d_j(a)>d^j(\hat{a})$ then either*

$$a\equiv R_0-R_j,\ A_j\not\equiv 0,\ \text{and } A_j\not\equiv -1$$

or

$$j<z,\ a\equiv R_0+S_{j+1},\ A_{j+1}\not\equiv 0,\ \text{and } A_j\not\equiv -1.$$

Moreover, if $d_j(a)=d^j(\hat{a})+2$ then both of the above conditions hold.

Proof. If $d_j(a)>0$ then $d^j(\hat{a})\geq 0$ by the previous lemma. So $d_j(a)\leq 0$. The remainder of the lemma follows from Corollary 6.7. □

Lemma 7.3. *If*

$$a\equiv R_0+S_j,\ A_j\not\equiv 0,\ A_j\not\equiv -1$$

then $d_j(a)<0$ and either

$$d^j(\hat{a})>d_j(a)$$

or

$$d^j(\hat{a})=d_j(a),\ j<z,\ a\equiv R_0+S_{j+1},\ A_{j+1}\not\equiv 0,\ A_{j+1}\not\equiv -1.$$

Proof. We have $a \equiv R_0 + S_j = R_0 + A_j + \varepsilon_j - R_j = R_0 - R_j + A_j + 1$, with $A_j \not\equiv 0$. So by Lemmas 2.8(iii) and 2.9, we have $d_j(a) < 0$. The remainder follows from Corollary 6.7. □

Lemma 7.4. *Let $j > 0$, $a \equiv R_0 - R_j$, $A_j \not\equiv 0$, and $A_j \not\equiv -1$. Then one of the following occurs.*

i. $d_{j-1}(a) < 0$ *and either*

$$d^{j-1}(\hat{a}) > d_{j-1}(a)$$

or

$$d^{j-1}(\hat{a}) = d_{j-1}(a),\ a \equiv R_0 - R_{j-1},\ A_{j-1} \not\equiv 0,\ \textit{and}\ A_{j-1} \not\equiv -1.$$

ii. $d_{j-1}(a) \geq 0$ *and* $d^{j-1}(\hat{a}) > d_{j-1}(a)$.

Proof. By Corollary 6.7, we have $d^{j-1}(\hat{a}) = d_{j-1}(a) + x$ where $x = 0$ if $a \equiv R_0 - R_{j-1}$, $A_{j-1} \not\equiv 0$, $A_{j-1} \not\equiv -1$ and $x > 0$ otherwise. If $d_{j-1}(a) < 0$ we get (i). Let $d_{j-1}(a) \geq 0$. We have to show that $x > 0$.

If on the contrary $x = 0$, then $a \equiv R_0 - R_{j-1}$, $A_{j-1} \not\equiv 0$, and $A_{j-1} \not\equiv -1$. Since by assumption $a \equiv R_0 - R_j$, we get $R_{j-1} - R_j \equiv 0$. Thus $A_{j-1} \not\equiv 0$, $A_{j-1} \not\equiv R_{j-1} - R_j$, $A_{j-1} \not\equiv R_{j-1} - R_j - 1$, and $a \equiv R_0 - R_j$. It follows from 2.8(ii) that $d_{j-1}(a) = -1$, giving a contradiction. □

Lemma 7.5. *Let*

$$j > 0,\ a \equiv R_0 + S_{j-1},\ A_{j-1} \not\equiv 0,\ A_{j-1} \not\equiv -1$$

and

$$a \equiv R_0 - R_j,\ A_j \not\equiv 0,\ A_j \not\equiv -1.$$

Then $d_{j-1}(a) = -2$ and $d^{j-1}(\hat{a}) = 0$.

Proof. It follows from Corollary 6.7 that $d^{j-1}(\hat{a}) = d_{j-1}(a) + 2$. If $d_{j-1}(a) > -2$ then $d^{j-1}(\hat{a}) > 0$. But by Lemma 7.1 (applied to $m(\lambda)$ in place of λ), $d^{j-1}(\hat{a}) > 0$ implies $d_{j-1}(a) \geq d^{j-1}(\hat{a}) - 1$, giving a contradiction. □

In the following two propositions we write d_j for $d_j(a)$ and d^j for $d^j(\hat{a})$.

Proposition 7.6. *Let $d_0 > 0$ and*

$$\{j_1 < \cdots < j_f\} = \{j \in [0,z] \mid d^j < d_j\}.$$

For every $t \in [1,f]$, there exist either x_t or y_t or both such that

i. $0 \leq x_t < j_t$, $d^{x_t} > d_{x_t}$, $d^{\nu} = d_{\nu}$ *for all* $\nu \in (x_t, j_t)$;

ii. $j_t < y_t \leq z$, $d^{y_t} > d_{y_t}$, $d^{\nu} = d_{\nu}$ *for all* $\nu \in (j_t, y_t)$, *and* $d_{\nu} < 0$ *for all* $\nu \in (j_t, y_t]$;

iii. *If $d^{j_t} = d_{j_t} - 2$ then both x_t and y_t exist;*

iv. *If y_t exists and $t < f$ then $y_t < j_{t+1}$. If x_t exists and $t > 0$ then $x_t > j_{t-1}$.*

v. *If y_t and x_{t+1} both exist then $y_t \leq x_{t+1}$, and $y_t = x_{t+1}$ implies $d_{y_t} = -2$, $d^{y_t} = 0$.*

Proof. Let $t \in [1, f]$. If $d_{j_t} > 0$ then by Lemma 7.1,

$$d^{j_t} = d_{j_t} - 1,\ j_t < z,\ a \equiv R_0 + S_{j_t+1},\ A_{j_t+1} \not\equiv 0, \text{ and } A_{j_t+1} \not\equiv -1.$$

So Lemma 7.3 implies that $d_{j_t+1} < 0$ and either

$$d^{j_t+1} > d_{j_t+1},$$

in which case we put $y_t = j_t + 1$, or

$$d^{j_t+1} = d_{j_t+1},\ j_t + 1 < z,\ a \equiv R_0 + S_{j_t+2},\ A_{j_t+2} \not\equiv 0, \text{ and } A_{j_t+2} \not\equiv -1.$$

Applying Lemma 7.3 repeatedly we find an integer y_t such that

$$\begin{aligned} &j_t < y_t \le z,\ a \equiv R_0 + S_v,\ A_v \not\equiv 0,\ A_v \not\equiv -1,\ d_v < 0 \\ &\text{for all } v \in (j_t, y_t],\ d^v = d_v \text{ for all } v \in (j_t, y_t), \text{ and } d^{y_t} > d_{y_t}. \end{aligned} \tag{8}$$

Now assume $d_{j_t} \le 0$. Then $j_t > 0$ (since $d_0 > 0$ by assumption), and $d^{j_t} < 0$. By Lemma 7.2, either

$$a \equiv R_0 - R_{j_t},\ A_{j_t} \not\equiv 0,\ A_{j_t} \not\equiv -1$$

or

$$j_t < z,\ a \equiv R_0 + S_{j_t+1},\ A_{j_t+1} \not\equiv 0,\ A_{j_t+1} \not\equiv -1,$$

and both of these conditions hold if $d^{j_t} = d_{j_t} - 2$.

If the latter condition holds then applying Lemma 7.3 repeatedly as above we show that (8) holds.

If the former condition is true then by Lemma 7.4 we have either

$$d^{j_t-1} > d_{j_t-1},$$

in which case we put $x_t = j_t - 1$, or

$$d^{j_t-1} = d_{j_t-1} < 0,\ a \equiv R_0 - R_{j_t-1},\ A_{j_t-1} \not\equiv 0, \text{ and } A_{j_t-1} \not\equiv -1.$$

The assumption $d_0 > 0$ now implies $j_t - 1 > 0$. Applying these arguments repeatedly we find an integer $x_t \ge 0$ such that

$$\begin{aligned} &a \equiv R_0 - R_v,\ A_v \not\equiv 0,\ A_v \not\equiv -1 \text{ for all } v \in (x_t, j_t], \\ &d^v = d_v \text{ for all } v \in (x_t, j_t), \text{ and } d^{x_t} > d_{x_t}. \end{aligned} \tag{9}$$

Assume that y_t exists for some $t < f$. Since $d^v = d_v$ for any $v \in (j_t, y_t)$ and $d^{y_t} > d_{y_t}$, we get $y_t < j_{t+1}$. Similarly $x_t > j_{t-1}$.

Assume that both y_t and x_{t+1} exist. Then $d^v = d_v$ for any $v \in (j_t, y_t) \cup (x_{t+1}, j_{t+1})$, $d^{y_t} > d_{y_t}$, and $d^{x_t} > d_{x_t}$ imply $y_t \le x_{t+1}$.

Finally notice that if $y_t = x_{t+1} =: u$ then it follows from (8) and (9) that $a \equiv R_0 + S_u,\ A_u \not\equiv 0,\ A_u \not\equiv -1$ and $a \equiv R_0 - R_{u+1},\ A_{u+1} \not\equiv 0,\ A_{u+1} \not\equiv -1$. So it suffices to apply Lemma 7.5 to complete the proof of the proposition. □

Proposition 7.7. *Assume that* $S_h := d_0 + d_1 + \cdots + d_h > 0$ *for any* $h \in [0, z]$. *Then* $S^h := d^0 + d^1 + \cdots + d^h > 0$ *for any* $h \in [0, z]$.

Proof. Since $d_0 = S_0 > 0$ we may apply Proposition 7.6. Let

$$\{j_1 < \cdots < j_f\} = \{j \in [0,z] \mid d^j < d_j\}$$

and x_t, y_t be as in 7.6.

First, we prove by induction on h the following intermediate fact.

($*$) $S^h \geq S_h - 1$, and if $S^h = S_h - 1$ then $h \in [j_t, y_t)$ for some t such that y_t exists.

If $h = 0$ and $d^0 = S^0 < S_0 = d_0$ then Lemma 7.1 implies $S^0 = S_0 - 1$. Moreover, we have $j_1 = 0$ and, by Proposition 7.6, y_1 exists.

Let $h > 0$ and assume that $S^h < S_h$.

If $h \notin \{j_1, \ldots, j_f\}$, i.e. $d^h \geq d_h$, then $S^{h-1} < S_{h-1}$. By the inductive hypothesis we have $S^{h-1} = S_{h-1} - 1$, and $h - 1 \in [j_t, y_t)$ for some t such that y_t exists. If $h - 1 < y_t - 1$ then $h \in (j_t, y_t)$, hence $d^h = d_h$ and $S^h = S_h - 1$, $h \in [j_t, y_t)$, as desired. If $h - 1 = y_t - 1$ then $h = y_t$, $d^h > d_h$, and $S^h \geq S_h$, contrary to our assumption that $S^h < S_h$.

Now assume that $h = j_t$ for some $t \in [1, f]$.

We first show that if x_t exists then $S^{h-1} > S_{h-1}$. Indeed, we prove first that $S^{x_t} > S_{x_t}$. By Proposition 7.6, we have that one of the following holds:

- y_{t-1} does not exist;
- $y_{t-1} < x_t$; or
- $y_{t-1} = x_t$, $d^{x_t} = 0$, $d_{x_t} = -2$.

In the first two cases $x_{t-1} \notin [j_v, y_v)$ for any v in view of 7.6. So by the inductive hypothesis we get $S^{x_t - 1} = S_{x_t - 1}$ or $x_t = 0$. So $d^{x_t} > d_{x_t}$ implies $S^{x_t} > S_{x_t}$. In the third case by the inductive hypothesis we get $S^{x_t - 1} \geq S_{x_t - 1} - 1$ or $x_t = 0$. So $d^{x_t} = d_{x_t} + 2$ implies $S^{x_t} > S_{x_t}$ again. Now, since $d^v = d_v$ for any $v \in (x_t, j_t)$ we conclude that $S^{h-1} > S_{h-1}$.

If $d^{j_t} = d_{j_t} - 2$ then by Proposition 7.6 both x_t and y_t exist and it remains to notice that $S^h = S^{h-1} + d^{j_t} \geq S_{h-1} + 1 + d_{j_t} - 2 = S_h - 1$.

Let $d^{j_t} \geq d_{j_t} - 1$. If x_t exists then it follows from $S^{h-1} > S_{h-1}$ that $S^h \geq S_h$. Assume that x_t does not exist. Then by Proposition 7.6 y_t exists, and we have only to prove that $S^h \geq S_h - 1$. Note that $S^{h-1} \geq S_{h-1}$ by the inductive hypothesis since $h - 1 \notin [j_v, y_v)$ for any v such that y_v exists. Indeed, if on the contrary $h - 1 \in [j_v, y_v)$ then $j_t = h \in (j_v, y_v]$ whence $d^{j_t} \geq d_{j_t}$, giving a contradiction. Now $S^h \geq S_h - 1$ follows.

Thus ($*$) is completely proved.

Now assume that there exists some h with $S^h \leq 0$. If $S^h < 0$ then $S_h \leq S^h + 1$ implies $S_h \leq 0$, giving a contradiction. So $S^h = 0$, $S_h = 1$, and $h \in [j_t, y_t)$ for some t such that y_t exists. Then by Proposition 7.6(ii), $h + 1 \leq z$, $d_{h+1} < 0$, whence $S_{h+1} \leq 0$, again giving a contradiction. □

Corollary 7.8. *Runner a of λ contains a degenerate good bead if and only if runner $\hat{a}$ of $m(\lambda)$ contains a degenerate good bead.*

Proof. It follows from Proposition 7.7 that $\sum_{i=0}^{h} d_i(a) > 0$ for any $h \in [0,z]$ if and only if $\sum_{i=0}^{h} d^i(\hat{a}) > 0$ for any $h \in [0,z]$. Now it suffices to apply Corollary 5.8. □

8. PROOF OF THE MAIN THEOREM

In this section we first translate some of the results obtained above from the language of abaci to the language of Young diagrams, and then prove the main theorem.

We recall some notions from [17]:

For a partition $\gamma = (l_1 \geq l_2 \geq \cdots \geq l_m > 0)$ we consider its Young diagram as the subset

$$\gamma = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i,\ 1 \leq j \leq l_i\} \subseteq \mathbb{Z} \times \mathbb{Z},$$

where we imagine the i-axis increasing downward and the j-axis increasing to the right.

All elements $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ are called *nodes*, while the elements $(i,j) \in \gamma$ are called *nodes of the diagram* γ.

If $A = (i,j)$ is a node we denote the residue class $(j-i) \pmod p$ by $\operatorname{res} A$ and call it the *residue* of A.

If $A = (i,j)$ and $B = (i',j')$ are nodes we say that B is *above* A (or that A is *below* B) if and only if $i' < i$. In this case we write $A \nearrow B$.

Young diagrams μ and ν are said to have the same *residue content* if and only if for any residue α modulo p the number of nodes of residue α in μ is equal to the number of nodes of residue α in ν.

If $l_i > l_{i+1}$ then the node (i,l_i) is called a *removable* node for γ (l_{m+1} is interpreted as 0).

We call (i,j) an *indent* node for γ if $i = 1$ and $j = l_1 + 1$, or if $i > 1$, $l_i < l_{i-1}$, and $j = l_i + 1$.

If $A = (i,l_i)$ is a removable node for γ we denote by γ_A the partition $(l_1, \dots, l_{i-1}, l_i - 1, l_{i+1}, \dots)$ of $n-1$ whose Young diagram of is $\gamma \setminus \{A\}$.

Definition 8.1. Let A be a removable node for γ. Set

$$M_A = \{B \mid B \text{ is an indent node for } \gamma,\ A \nearrow B,\ \operatorname{res} B = \operatorname{res} A\}.$$

We call A *normal* if and only if for any $B \in M_A$ there exists a removable node $C(B)$ such that the following three conditions hold:

(1) $A \nearrow C(B) \nearrow B$;
(2) $\operatorname{res} C(B) = \operatorname{res} A$;
(3) $|\{C(B) \mid B \in M_A\}| = |M_A|$.

We call the node A *good* if it is normal and $A \nearrow D$ for any normal node D with $\operatorname{res} D = \operatorname{res} A$.

We call a good node A for γ *degenerate* if and only if $(\lambda_A)^{(1)} = \lambda^{(1)}$. Otherwise we call it *non-degenerate*.

Lemma 8.2. *Let γ be a p-regular partition and let Γ be an abacus for γ with R beads. Assume $B_1 \nearrow \dots \nearrow B_q$ are all the removable nodes for γ and $C_1 \nearrow \dots \nearrow C_q$*

are the top q indent nodes for γ *(there are* $q+1$ *indent nodes for* γ*). Then there exist q ends and q beginnings for* Γ.

Moreover, let $k_1,\dots,k_q$ *and* $l_1,\dots,l_q$ *be all the beginnings and ends, respectively, for* Γ*, with* $P_\Gamma(k_i) < P_\Gamma(k_{i+1})$ *and* $P_\Gamma(l_i) < P_\Gamma(l_{i+1})$ *for* $i \in [1,q)$*. Then for any* $i \in [1,q]$ *the following hold.*

i. $\Gamma(k_i)$ *is an abacus for* γ_{B_i}*;*
ii. $k_i \in (a)_\Gamma$ *if and only if* $\operatorname{res} B_i = a - R$*;*
iii. $l_i \in (a-1)_\Gamma$ *if and only if* $\operatorname{res} C_i = a - R$*;*
iv. k_i *is a normal bead for* Γ *if and only if* B_i *is a normal node for* γ*;*
v. k_i *is a good bead for* Γ *if and only if* B_i *is a good node for* γ*;*
vi. k_i *is a degenerate good bead for* Γ *if and only if* B_i *is a degenerate good node for* γ.

Proof. If we represent γ in the canonical form (1) from the introduction, then $k = q$ and we can write

$$\gamma = (\nu_1^{\alpha_1},\dots,\nu_q^{\alpha_q}),\ \nu_1 > \cdots > \nu_q > 0,\ \alpha_i > 0.$$

The numbers of beginnings and ends for Γ are both equal to the number of strings of proper beads in Γ which in turn equals q.

Part (i) follows immediately from the definitions.

Note that

$$B_{q+1-i} = (\alpha_1 + \cdots + \alpha_i, \nu_i)$$

and

$$\begin{aligned} P_\Gamma(k_{q+1-i}) &= (R - (\alpha_1 + \cdots + \alpha_q) - 1) + \nu_i + (\alpha_{i+1} + \cdots + \alpha_q) + 1 \\ &= R + \nu_i - (\alpha_1 + \cdots + \alpha_i). \end{aligned}$$

This implies (ii), and (iii) is obtained similarly. The remaining parts of the lemma follow from (i)–(iii) and the definitions. □

Let λ be a p-regular partition of n. Recall the Mullineux symbols

$$G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix} \text{ and } G_p(m_n(\lambda)) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ S_0 & S_1 & \dots & S_z \end{pmatrix}.$$

Lemma 8.3. i. *If A is a non-degenerate good node for* λ *of residue* α *then there exists a good node C for* $\lambda^{(1)}$*, with* $\operatorname{res} C = \alpha$ *and* $(\lambda_A)^{(1)} = (\lambda^{(1)})_C$*. Moreover, in this case the first column of the Mullineux symbol* $G_p(\lambda_A)$ *is* $\begin{pmatrix} A_0 \\ R_0 \end{pmatrix}$.

ii. *There exists a degenerate good node of residue* α *for* λ *if and only if there exists a degenerate good node of residue* $-\alpha$ *for* $m(\lambda)$.

iii. *If A is a degenerate good node for* λ *then the first column of the Mullineux symbol* $G_p(\lambda_A)$ *is equal to* $\begin{pmatrix} A_0 - 1 \\ R_0 - \xi \end{pmatrix}$ *where* $\xi = 0$ *or* $\xi = 1$.

iv. *Let A be a degenerate good node for* λ *and* ξ *be defined as in* (iii)*. Then* $A_0 \equiv 1$ *implies* $\xi = 1$*, and* $A_0 \equiv 0$ *implies* $\xi = 0$.

v. *If there exists a good node of residue* α *for* $\lambda^{(1)}$ *then there exists a good node of residue* α *for* λ.

Proof. According to Lemma 8.2, there exists a good node of residue α for λ (resp., $\lambda^{(1)}$) if and only if there exists a good bead in runner $a = \alpha + R_0$ of the canonical abacus for λ (resp., abacus with R_0 beads for $\lambda^{(1)}$). By the same lemma, this good node is degenerate if and only if the corresponding good bead is so.

Similarly, to a (degenerate) good node of residue $-\alpha$ for $m(\lambda)$ there corresponds a (degenerate) good bead in runner $-\alpha + S_0$ of the canonical abacus for $m(\lambda)$.

Notice that $\hat{a} \equiv R_0 + S_0 - a \equiv -\alpha + S_0$. Now part (i) follows from Lemma 4.3(i) and (iii); part (ii), from Corollary 7.8; part (iii), from Lemma 4.3(iii); part (iv), from Lemma 4.3(iv); and, finally, part (v) follows from Corollary 3.3. □

In the proof of 8.7 we shall need the following three results.

Lemma 8.4 ([12, 2.7.41])**.** *Suppose that* ν *and* μ *are partitions of the same integer. Then* $\mathrm{core}(\nu) = \mathrm{core}(\mu)$ *if and only if the Young diagrams of* ν *and* μ *have the same residue content.*

Lemma 8.5. *Let* ν *and* μ *be* p*-regular partitions with*

$$G_p(\nu) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}, \quad G_p(\mu) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 - 1 & R_1 & \dots & R_z \end{pmatrix}.$$

If $A_0 \not\equiv 0$ *then* $\mathrm{core}(\nu) \neq \mathrm{core}(\mu)$.

Proof. We have $\nu^{(1)} = \mu^{(1)}$. Since $A_0 \not\equiv 0$ we get from 1.1 that $R_0 - 1 - R_1 > 0$, or $R_0 \geq R_1 + 2$. Let $A_0 = pd + r$, with $d, r \in \mathbb{Z}, r \in (0, p)$.

Any p-segment of length p contains exactly one node of every residue. Moreover, the bottom p-segments of both ν and μ contain r nodes. Since $R_0 \geq R_1 + 2$, the bottom p-segment of ν contains nodes of residues $-(R_0 - 1) + x,\ x \in [0, r)$, while the bottom p-segment of μ contains nodes of residues $-(R_0 - 2) + x,\ x \in [0, r)$. Since $0 < r < p$, we conclude that ν and μ have distinct residue contents. An application of Lemma 8.4 completes the proof. □

Lemma 8.6. *Let* γ *be a* p*-regular partition, and* α *be a residue modulo* p*. Then the number of nodes in* γ *of residue* α *is equal to the number of the nodes in* $m(\gamma)$ *of residue* $-\alpha$*.*

Proof. By Proposition 6.2, $\mathrm{core}(m(\gamma)) = (\mathrm{core}(\gamma))'$. So the number of the nodes of residue $-\alpha$ in $\mathrm{core}(m(\gamma))$ is equal to the number of the nodes of residue α in $\mathrm{core}(\gamma)$. Now it remains to notice that any rim p-hook contains exactly one node of every residue. □

Notation. If $\begin{pmatrix} B_0 & B_1 & \dots & B_z \\ T_0 & T_1 & \dots & T_z \end{pmatrix}$ is the Mullineux symbol of a p-regular partition γ, then for any $i \in [0, z]$ we write:

$$\varepsilon(B_i) = \begin{cases} 0, & \text{if } p | B_i, \\ 1, & \text{otherwise;} \end{cases}$$

$$f(B_i, T_i) = B_i + \varepsilon(B_i) - T_i.$$

Then by definition

$$\begin{pmatrix} B_0 & B_1 & \dots & B_z \\ f(B_0,T_0) & f(B_1,T_1) & \dots & f(B_z,T_z) \end{pmatrix}$$

is the Mullineux symbol for $m(\gamma)$.

Theorem 8.7. *Suppose A is a good node for a p-regular partition λ of n, with $\operatorname{res} A = \alpha$. Then there exists a good node B for $m(\lambda)$ with $\operatorname{res} B = -\alpha$, and $m(\lambda_A) = m(\lambda)_B$.*

Proof. We prove the theorem by induction on n. If $n = 1$, it is trivial. Assume $n > 1$ and the theorem has been proved for all $n' < n$. Put

$$G_p(\lambda) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ R_0 & R_1 & \dots & R_z \end{pmatrix}, \quad G_p(\lambda_A) = \begin{pmatrix} B_0 & B_1 & \dots & B_x \\ T_0 & T_1 & \dots & T_x \end{pmatrix}.$$

Then

$$G_p(m(\lambda)) = \begin{pmatrix} A_0 & A_1 & \dots & A_z \\ S_0 & S_1 & \dots & S_z \end{pmatrix}, \quad G_p(m(\lambda_A)) = \begin{pmatrix} B_0 & B_1 & \dots & B_x \\ U_0 & U_1 & \dots & U_x \end{pmatrix}$$

where $S_i = f(A_i, R_i)$ for $i \in [0,z]$, and $U_j = f(B_j, T_j)$ for $j \in [0,x]$.

Suppose first that A is non-degenerate. Then, in view of Lemma 8.3(i), there is a good node C for $\lambda^{(1)}$, with $\operatorname{res} C = \alpha$ and $(\lambda_A)^{(1)} = (\lambda^{(1)})_C$. By the inductive hypothesis there is a good node D for $m(\lambda^{(1)}) = m(\lambda)^{(1)}$, with $\operatorname{res} D = -\alpha$ and $m((\lambda^{(1)})_C) = m(\lambda^{(1)})_D$. According to Lemma 8.3(v), there is a good node B for $m(\lambda)$ with $\operatorname{res} B = -\alpha$. In view of 8.3(ii), B is non-degenerate. Therefore $(m(\lambda)_B)^{(1)} = (m(\lambda)^{(1)})_D$. Let

$$G_p(m(\lambda)_B) = \begin{pmatrix} C_0 & C_1 & \dots & C_y \\ V_0 & V_1 & \dots & V_y \end{pmatrix}.$$

Since

$$m((\lambda_A)^{(1)}) = m((\lambda^{(1)})_C) = m(\lambda^{(1)})_D = (m(\lambda)_B)^{(1)},$$

we have $x = y$ and $B_i = C_i$, $U_i = V_i$ for $i \in [1,x]$. By Lemma 8.3(i), $B_0 = A_0$, $T_0 = R_0$ and $C_0 = A_0$, $V_0 = S_0$. So $B_0 = C_0$. It suffices to use the equalities $U_0 = f(B_0, T_0)$ and $S_0 = f(A_0, R_0)$ to prove that $U_0 = V_0$. Thus $G_p(m(\lambda)_B) = G_p(m(\lambda_A))$, hence $m(\lambda)_B = m(\lambda_A)$.

Assume that A is degenerate. By Lemma 8.3(ii), there is a degenerate good node B for $m(\lambda)$ with $\operatorname{res} B = -\alpha$.

By definition, $(\lambda_A)^{(1)} = (\lambda)^{(1)}$. So $x = z$, $B_i = A_i$, and $T_i = R_i$ for $i \in [1,z]$. Moreover, it follows from Lemma 8.3, parts (iii) and (iv), that $B_0 = A_0 - 1$, $T_0 = R_0 - \xi$ where $\xi \in \{0,1\}$, and $A_0 \equiv 1$ implies $\xi = 1$, $A_0 \equiv 0$ implies $\xi = 0$. Thus,

$$G_p(m(\lambda_A)) = \begin{pmatrix} A_0 - 1 & A_1 & \dots & A_z \\ f(A_0 - 1, R_0 - \xi) & S_1 & \dots & S_z \end{pmatrix}.$$

Similarly

$$G_p(m(\lambda)_B) = \begin{pmatrix} A_0 - 1 & A_1 & \dots & A_z \\ S_0 - \eta & S_1 & \dots & S_z \end{pmatrix}$$

where $\eta \in \{0,1\}$, and $A_0 \equiv 1$ implies $\eta = 1$, $A_0 \equiv 0$ implies $\eta = 0$.

We consider three cases.

(a) $A_0 \equiv 0$. Then

$$\xi = 0,\ \eta = 0;$$

$$S_0 = f(A_0, R_0) = A_0 + \varepsilon(A_0) - R_0 = A_0 - R_0;$$

$$\begin{aligned} f(A_0 - 1, R_0 - \xi) &= f(A_0 - 1, R_0) = A_0 - 1 + \varepsilon(A_0 - 1) - R_0 \\ &= A_0 - 1 + 1 - R_0 = A_0 - R_0 = S_0 - \eta. \end{aligned}$$

Thus $G_p(m(\lambda)_B) = G_p(m(\lambda_A))$, i.e. $m(\lambda)_B = m(\lambda_A)$.

(b) $A_0 \equiv 1$. Then

$$\xi = 1,\ \eta = 1;$$

$$S_0 = f(A_0, R_0) = A_0 + \varepsilon(A_0) - R_0 = A_0 + 1 - R_0;$$

$$\begin{aligned} f(A_0 - 1, R_0 - \xi) &= f(A_0 - 1, R_0 - 1) = A_0 - 1 + \varepsilon(A_0 - 1) - (R_0 - 1) \\ &= A_0 - 1 - R_0 + 1 = A_0 - R_0 = S_0 - \eta. \end{aligned}$$

Thus $G_p(m(\lambda)_B) = G_p(m(\lambda_A))$, i.e. $m(\lambda)_B = m(\lambda_A)$.

(c) $A_0 \not\equiv 0$, $A_0 \not\equiv 1$. Then

$$S_0 = f(A_0, R_0) = A_0 + \varepsilon(A_0) - R_0 = A_0 + 1 - R_0;$$

$$\begin{aligned} f(A_0 - 1, R_0 - \xi) &= f(A_0 - 1, R_0 - \xi) = A_0 - 1 + \varepsilon(A_0 - 1) - (R_0 - \xi) \\ &= A_0 - 1 + 1 - R_0 + \xi = A_0 - R_0 + \xi. \end{aligned}$$

Thus

$$S_0 - \eta = f(A_0 - 1, R_0 - \xi) + 1 - \eta - \xi,$$

which gives

$$f(A_0 - 1, R_0 - \xi) - 1 \leq S_0 - \eta \leq f(A_0 - 1, R_0 - \xi) + 1.$$

We have $\mathrm{core}(m(\lambda_A)) = \mathrm{core}(m(\lambda)_B)$ in view of Lemmas 8.4 and 8.6. Now Lemma 8.5 implies $S_0 - \eta = f(A_0 - 1, R_0 - \xi)$, i.e. $G_p(m(\lambda)_B) = G_p(m(\lambda_A))$, or $m(\lambda)_B = m(\lambda_A)$. □

Thus we have proved Conjecture 3. Now the Main Theorem follows from Theorem 4 (cf. Introduction).

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