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OVERGROUPS OF IRREDUCIBLE LINEAR GROUPS, II

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ABSTRACT. Determining the subgroup structure of algebraic groups (over an algebraically closed field K of arbitrary characteristic) often requires an understanding of those instances when a group Y and a closed subgroup G both act irreducibly on some module V, which is rational for G and Y. In this paper and [4], we give a classification of all such triples (G,Y,V) when G is a non-connected algebraic group with simple identity component X, V is an irreducible G-module with restricted X-high weight(s), and Y is a simple algebraic group of classical type over K sitting strictly between X and $\mathrm{SL}(V)$.

CONTENTS

1. INTRODUCTION

E. B. Dynkin in 1957 [3, 2] classified the maximal closed connected subgroups of simple algebraic groups when the underlying (algebraically closed) field has characteristic 0. Seitz ([9, 10]) and Testerman ([13]) completed the same program in positive characteristic in the 1980's. Their analyses for the classical group cases were based primarily on a striking result: If G is a simple algebraic group and $\varphi : G \to \mathrm{SL}(V)$ is a tensor indecomposable irreducible rational representation, then with specified exceptions the image of G is maximal among closed connected subgroups of one of the classical groups $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SO}(V)$. What is most striking is the brevity of the list of exceptions.

From a slightly different perspective, the question these authors answered was: Given a closed, connected subgroup G of $\mathrm{SL}(V)$ for some vector space V, with G acting irreducibly on V, find all possibilities for closed, connected overgroups Y of G in $\mathrm{SL}(V)$.

This question of irreducible overgroups appears in other contexts as well, sometimes for non-connected subgroups. Here and in [4] we present results for some such non-connected subgroups; namely, those with simple identity components. The overall program is to classify all possible triples (G,Y,V) with G and Y both closed subgroups of $\mathrm{SL}(V)$ acting

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irreducibly on V, $G < \mathrm{Aut}(Y)$, $Y \neq \mathrm{SL}(V), \mathrm{SO}(V)$, or $\mathrm{Sp}(V)$, and Y a simple group of classical type (the corresponding question for Y of exceptional type is also open). We give complete results for the case when G is not connected but has simple identity component X, and the T_Y-high weight and T_X-high weights of V are restricted. Specifically, the papers are concerned with the proof of Theorem 1.

Theorem 1. *Let G be a non-connected algebraic group, over a field K of arbitrary characteristic $p \geq 0$, with simple identity component X. Let V be an irreducible KG-module with restricted X-high weight(s). Let Y be a simple algebraic group of classical type such that $X < Y < \mathrm{SL}(V)$ and $G \leq \mathrm{Aut}(Y)$. Then $V|_Y$ is irreducible with restricted high weight if and only if $Y = \mathrm{SO}(V)$, $Y = \mathrm{Sp}(V)$, or (X,Y,V) appears in Table 1 (page 41) or Table 2 (page 42).*

If G has simple identity component X, then $G \leq \mathrm{Aut}(X)$. Since we require that $G \neq X$, we therefore may restrict our attention to X of type A_m, D_m, or E_6. We assume henceforth that Y is simply connected, and that X and Y act on W, the natural module for Y.

The analysis is different depending on whether X acts reducibly or irreducibly on W. We settled the reducible case in [4], and we consider the irreducible case here. Also, we will assume here that the involutory graph automorphism of X, if it is in G, also acts on W (though it need not be in Y), as we dealt with the case when it does not act on W in the final section of [4].

If $V|_X$ is irreducible, then we are in the case examined by Seitz in [9], with the additional condition that X have an outer automorphism which acts on V. We examine Table 1 of that paper, and find that we have such a situation in the examples there labelled I_4, I_5, I_6 for $n = 3$, II_1, S_1, S_8 (in S_8 we could take $G = X\langle t\rangle$, $G = X\langle s\rangle$, or $G = X\langle s,t\rangle$, where t,s are outer automorphisms of X of order 2 and 3 respectively), and MR_4. These examples are collected in Table 1, and henceforth we shall assume that $V|_X$ is reducible.

1.1. **Notation and Conventions.** All structures are assumed to be constructed over the same algebraically closed field K, of characteristic $p \geq 0$. Throughout, X will denote a simple algebraic group over K admitting an outer automorphism (so X is of type A_m, D_m, or E_6). A fixed standard graph automorphism of order 2 will be denoted by t, and if X has an outer automorphism of order 3 (i.e. if $X = D_4$), we will fix one and denote it by s. Thus G is $X\langle t\rangle$ except possibly when $X = D_4$, in which case we also consider $G = X\langle s\rangle$ and $G = X\langle s,t\rangle$.

For any reductive group H we consider, with fixed maximal torus T, $\Sigma(H)$ will denote the roots of H relative to T. If $\gamma \in \Sigma(H)$, we let $h_\gamma : K^* \to T$ be the one-parameter subgroup of T such that $\alpha(h_\gamma(x)) = x^{<\alpha,\gamma>}$ for any $\alpha \in \Sigma(H)$ and $x \in K^*$.

We let B_X be a fixed t-stable Borel subgroup of X, containing a fixed t-stable maximal torus T_X. Define sets of simple roots $\{\beta_1, \beta_2, ..., \beta_m\} = \Pi(X) \subseteq \Sigma(X)$ and fundamental dominant weights $\{\delta_1, \dots, \delta_m\}$ with respect to T_X and B_X, but with the opposite of the standard convention: The set of positive roots $\Sigma^+(X)$ is defined by $B_X = U_X T_X$ where $U_X = \prod U_{-\alpha}$ for $\alpha \in \Sigma^+(X)$. Then for $J \subseteq \Pi(X)$, P_X is the opposite of the standard parabolic corresponding to J. We assume the δ_i are numbered so that δ_i corresponds to β_i for every i.

The group Y will be a simple algebraic group over K of classical type and rank n (A_n, B_n, C_n or D_n), such that $X < Y$ and $G \leq \mathrm{Aut}(Y)$. Let $\{\alpha_1, \alpha_2, ..., \alpha_n\} = \Pi(Y)$ be a set of simple roots of Y, and $\{\lambda_i\}$ the set of fundamental dominant weights such that λ_i corresponds to α_i. Notation and conventions similar to those used for X are used for parabolic subgroups of Y.

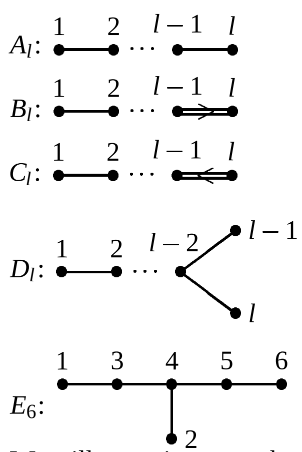
For a group H acting on a module M, $[M,H^l]$ will denote the l-fold commutator of H with M.

The K-vector space V is assumed to be a restricted irreducible Y-module with high weight $\lambda = \sum a_i \lambda_i$, such that V is irreducible as a G-module but not as an X-module (see the comment at the end of the previous subsection). We assume that the T_X-high weights of V are restricted as well. So if $G = X\langle t\rangle$, then $V|_X = V_1 \oplus V_2$, where each of V_1, V_2 is a restricted irreducible X-module.

The natural module for Y will be denoted by W. We assume that W is irreducible as an X-module, and δ will denote its T_X-high weight. As in [4], we will always assume that Y is the smallest of $\mathrm{SL}(W)$, $\mathrm{SO}(W)$, $\mathrm{Sp}(W)$ containing X.

Finally, we assume that G acts on W, as the case when it does not was considered in [4]

We label Dynkin diagrams for the groups we will be dealing with as follows, and we always number fundamental roots and fundamental dominant weights to agree with this labelling:



We will sometimes use the standard partial order on weights: $\nu \succ \mu$ if and only if $\nu - \mu$ is a sum of positive roots.

2. Q_X-Levels and Embeddings of Parabolics

In this section we introduce important facts about the "commutator series" of a module of a simple algebraic group.

Lemma 2.1. *If H is a simple algebraic group whose root system has only one root length, then restricted irreducible H-modules are tensor indecomposable (in particular, restricted irreducible X-modules are tensor indecomposable).*

Proof. This is part of 1.6 of [9]. □

Lemma 2.2. *Let M be an irreducible restricted H-module with high weight γ for some simple algebraic group H. Let P be a proper parabolic subgroup of H, with $P = QL$ a Levi decomposition. Then $M/[M,Q]$ is irreducible for L and for $L' = [L,L]$, with $T_{L'}$-high weight $\gamma|_{T_{L'}}$.*

Proof. This is 1.7 and 2.1 of [9]. □

Let H, M, γ, and P be as in the last lemma. Let $\{\varepsilon_i\}$ be the set of fundamental roots of H.

Definition 2.3. Let μ be a weight of M, say $\mu = \gamma - \sum c_i \varepsilon_i$, with each $c_i \geq 0$. The *Q-level* of μ is $\sum c_j$, where the sum ranges over those j for which $\varepsilon_j \in \Pi(H) - \Pi(L')$. The *Q-level* *l of M* is the sum of weight spaces for weights having Q-level l and is denoted M_l.

Lemma 2.4. *H, M, and P as above. If H is simply laced or if $p > 2$ ($p > 3$ for $H = G_2$), then*

1. $[M, Q^l] = \oplus M_\mu$*, the sum taken over those weights μ having Q-level at least l.*
2. $[M, Q^l]/[M, Q^{l+1}] \cong M_l$
3. $\dim([M, Q^l]/[M, Q^{l+1}]) \leq s \cdot \dim([M, Q^{l-1}]/[M, Q^l])$*, where s is the number of positive roots β such that $U_{-\beta} \leq Q$ and $\beta = \varepsilon_i + \beta'$ for some $\varepsilon_i \in \Pi(H) - \Pi(L')$, with $\beta' = 0$ or a sum of roots in $\Pi(L')$.*
4. $\dim([M, Q^l]/[M, Q^{l+1}]) \leq \dim(Q) \cdot \dim([M, Q^{l-1}]/[M, Q^l])$.

Proof. This is 2.3 of [9]. □

We will write $M^l(Q)$ for the quotient $[M, Q^{l-1}]/[M, Q^l]$.

Lemma 2.5. *Let $H = A_l$; let c be an integer such that $0 < c < p$; and let $\gamma_1, \ldots, \gamma_l$ be the fundamental dominant weights for H. The irreducible module M having high weight $c\gamma_1$ or $c\gamma_l$ has all weight spaces of dimension 1; in particular,* $\dim(M) = (l + c)!/l!c!$.

Proof. This is 1.14 of [9]. □

We will occasionally use the Weyl character formula for dimensions of Weyl modules.

Finally, it was shown in [4] that when X acts irreducibly on W, we may assume W is in fact restricted as an X-module:

Lemma 2.6. *If X acts irreducibly on W, then as an X-module, W has a restricted high weight.*

Now let P_X be a parabolic subgroup of X, and $P_X = Q_X L_X$ a Levi decomposition with $T_X \leq L_X$ (if P_X is t-stable, choose L_X to also be t-stable). Now X acts irreducibly on W with high weight δ, which is restricted by the Lemma above. We gave in [4] the following construction of a parabolic subgroup P_Y of Y (with $P_Y = Q_Y L_Y$ a Levi decomposition) such that $P_X \leq P_Y$, $Q_X \leq Q_Y$, $L_X \leq L_Y$. Let $Z = Z(L_X)^\circ$.

Lemma 2.7. *The stabilizer in Y of the commutator series*

$$W > [W, Q_X] > [W, Q_X, Q_X] > \cdots > 0$$

is a parabolic subgroup P_Y of Y satisfying the following:

1. $P_X \leq P_Y$ *and* $Q_X \leq Q_Y = R_u(P_Y)$.
2. $L_Y = C_Y(Z)$ *is a Levi factor of P_Y containing L_X.*
3. *If T_Y is a maximal torus of Y containing T_X, then $T_Y \leq L_Y$.*

Proof. This is 2.7 of [4]. □

We give more information about this embedding for particular groups X and parabolic subgroups P_X below and in subsequent sections. For the next Lemma, we assume that $t \in G$ (where t is the fixed outer automorphism of X) and $V|_X = V_1 \oplus V_2$, with V_1, V_2 irreducible X-modules. This Lemma was proved in [4, 2.8 and 2.9]:

Lemma 2.8. *If P_X is a t-stable parabolic subgroup of X and P_X is embedded in a parabolic subgroup P_Y of Y as above, then*

1. P_Y *is likewise t-stable;*
2. $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$ *as L_X-modules.*

Let $P_Y = L_Y Q_Y$ be a parabolic subgroup of Y. For each $\gamma \in \Pi(Y) - \Pi(L'_Y)$, we define a certain normal subgroup K_Y^γ of P_Y, as in [9, page 44]: Let $\Sigma_\gamma(Y)$ denote the set of roots in $\Sigma(Y)$ having γ-coefficient -1 and zero coefficient for other roots in $\Pi(Y) - \Pi(L'_Y)$. Then let K_Y^γ be the product of those T_Y-root subgroups U_β for $\beta \in \Sigma^-(Y) - \Sigma^-(L'_Y) - \Sigma_\gamma(Y)$. From the commutator relations it follows that K_Y^γ is normal in P_Y and we let $Q_Y^\gamma = Q_Y/K_Y^\gamma$. This construction also applies to a parabolic subgroup P_X of X. In particular, if P_X is a maximal parabolic subgroup corresponding to $\alpha \in \Pi(X)$, then set $Q_X^\alpha = Q_X/K^\alpha$, where K^α is the product of those T_X-root subgroups corresponding to roots having α-coefficient strictly less than -1.

The Lemma below will be used heavily in dimension arguments.

Lemma 2.9. *If $P_X = Q_X L_X$ is a maximal parabolic subgroup of X corresponding to $\alpha \in \Pi(X)$, and P_X is embedded in a parabolic subgroup P_Y of Y as in Lemma 2.7, then* $\dim(V^2(Q_Y)) \le \dim(Q_X^\alpha) \cdot \dim(V^1(Q_X))$.

Proof. This is part of Proposition 2.14 in [9]. □

Lemma 2.10. *If $P_X = Q_X L_X$ is a maximal parabolic subgroup corresponding to $\alpha \in \Pi(X)$, then:*

1. $K^\alpha = [Q_X, Q_X]$.
2. *Q_X^α is an irreducible L'_X-module with $-\alpha$ as its $T_{L'_X}$-high weight.*

Proof. See 3.2 in [9] (remembering that X is of type A_m, D_m, or E_6). □

Again assume $t \in G$. Let P_X be a parabolic subgroup of X (not necessarily t-stable) containing the fixed t-stable Borel subgroup B_X. Embed P_X in a parabolic subgroup P_Y of Y via the above construction. Write $L'_Y = L_1 \times \cdots \times L_r$, a direct product of simple groups. By Lemma 2.2, L'_Y acts irreducibly on $V^1(Q_Y) = V/[V, Q_Y]$. Then $V^1(Q_Y) = V^1 \otimes \cdots \otimes V^r$ where for each i, V^i is an irreducible module for L_i. The embedding $L_X \to L_Y$ gives an embedding of L'_X into $L_1 \times \cdots \times L_r$, and via the projections $L'_X \to L_i$, any L_i-module, in particular V^i, can be regarded as a module for L'_X.

Since $Q_X \le Q_Y$, we have $[V, Q_X] \le [V, Q_Y]$ and hence $V/[V, Q_Y]$ is a quotient of $V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$, with each of these summands irreducible L'_X-modules. Since $L'_X \le L'_Y$, this implies that either $V/[V, Q_Y]$ is irreducible for L'_X, or $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$. Lemma 2.8 tells us that the latter happens when P_X is t-stable. The following was proved in [4, 2.11]:

Lemma 2.11. *If V, $P_X = L_X Q_X$, $P_Y = L_Y Q_Y$, and L_i are as above with P_X t-stable, then only one L_i acts nontrivially on $V/[V, Q_Y]$.*

3. THE CASE $X = A_m$

As always, let $X < Y$ be simple algebraic groups over an algebraically closed field K of characteristic p ($= 0$ or a prime), with X admitting an involutory graph automorphism t which also acts on Y, and Y of classical type. Let $\{\beta_i\}$ ($\{\alpha_i\} = \Pi(Y)$) be the set of fundamental roots for X (Y); $\{\delta_i\}$ ($\{\lambda_i\}$) the corresponding fundamental dominant weights for X (Y). The fixed t-stable Borel subgroup B_X of X contains a t-stable maximal torus T_X. Let $V = V(\lambda)$ be a restricted irreducible t-stable Y-module with high weight $\lambda = \sum a_i \lambda_i$, such that $V|_{X\langle t\rangle}$ is irreducible, but $V|_X = V_1 \oplus V_2$, with V_1, V_2 restricted irreducible X-modules. We denote the T_X-high weight of V_1 by $b_1\delta_1 + b_2\delta_2 + \cdots + b_m\delta_m$, so the T_X-high weight of V_2 is $b_m\delta_1 + b_{m-1}\delta_2 + \cdots + b_1\delta_m$ The natural module for Y is W.

The main result of the section is:

Theorem 3.1. *If X acts irreducibly on W with high weight δ, t acts on W, and X is of type A_m, then $p \notin \{2,3,5,7\}$, $X = A_3$, $Y = D_{10}$, $\delta = 2\delta_2$, and the high weights of $V|_X, V|_Y$ are as in* U_5 *of Table 2.*

Notice that since t acts on W, the high weight $\delta = d_1\delta_1 + \cdots + d_m\delta_m$ of $W|_X$ must be symmetric, i.e. $d_1 = d_m$, $d_2 = d_{m-1}$, etc. But then by a result of Steinberg ([11, page 226]), X fixes a nondegenerate bilinear form on W; the form is orthogonal if $p \neq 2$.

The strategy we use to rule out most possibilities for the high weight δ is to show that the construction (outlined in Lemma 2.7) of a parabolic subgroup of Y containing the fixed (t-stable) Borel subgroup of X gives a contradiction in all but a few cases. After giving the Lemma which we usually use to produce the contradiction, we will treat the A_2 and A_3 cases first, followed by the general argument.

3.1. **Some Facts About P_Y.** We use the construction given in Lemma 2.7 of a parabolic subgroup $P_Y < Y$ containing the fixed t-stable Borel subgroup B_X. Namely, P_Y is taken to be the stabilizer in Y of the flag in W given by "U_X-levels."

We want to use Lemma 3.2 below to produce a contradiction in most cases; we will show that L_Y' has a factor of type A_1 only under strong conditions. Before proceeding with the general proof, we need some facts about the flag in W of which P_Y is the stabilizer.

Recall that for $i \geq 0$, $W_i = \sum_{e_1+e_2+\cdots=i} W_{\delta-e_1\beta_1-e_2\beta_2-\ldots}$, the sum taken over $e_j \geq 0$. Each space W_i is $T_X\langle t\rangle$-stable, and if $u \in U_{-\alpha}$, then

$$uW_{\delta-e_1\beta_1-e_2\beta_2-\ldots} \subseteq \sum_{m\geq 0} W_{\delta-e_1\beta_1-e_2\beta_2-\cdots-m\alpha}.$$

So $B_X\langle t\rangle$ stabilizes each factor

$$\left(\sum_{i\geq m} W_i\right) / \left(\sum_{i\geq m+1} W_i\right).$$

By Lemma 2.4, $W_i \cong [W, U_X^i]/[W, U_X^{i+1}]$.

Let l be minimal with respect to $[W, U_X^{l+1}] = 0$, and notice that l is then the level of the low weight $-\delta$. If $Y = \mathrm{Sp}(W)$ or $Y = \mathrm{SO}(W)$, with the form denoted by $(\ ,\ )$, then we noted in [4, proof of 2.7] that $(u,v) = 0$ for $u \in W_i$, $v \in W_j$ unless $i + j \leq l$. Thus the W_i for $i > l/2$, along with a maximal totally singular subspace of $W_{l/2}$ (if l is even), span a maximal totally singular subspace of W.

Let w^+ be an T_X-high weight vector of W. Then $W_0 = \langle w^+\rangle$, $W_l = \langle w_0 w^+\rangle$, and B_X is contained in the full stabilizer P_Y of the flag

$$W = \sum_{i\geq 0} W_i \geq \sum_{i\geq 1} W_i \geq \cdots \geq \sum_{i\geq l} W_i = \langle w_0 w^+\rangle \geq 0.$$

Let $P_Y = L_Y Q_Y$ be a Levi decomposition of P_Y; then if $u \in U_X, w \in W_m$, we have $uw - w \in \sum_{i\geq m+1} W_i$, so $U_X \leq Q_Y$. We have $T_X \leq L_Y = C_Y(Z)$ for $Z = Z(L_X)^\circ$.

Choose a basis for each W_i (with the basis for $W_{l/2}$ chosen maximally hyperbolic — note that $W_{l/2} \cong \left(\sum_{i\geq l/2} W_i\right) / \left(\sum_{i>l/2} W_i\right)$ is the only possible non-singular quotient in the flag); the union of these bases is a basis for W. With respect to this basis, L_Y' consists of block matrices, each block corresponding to W_i for some i. On the other hand, each W_i for $0 < i \leq l/2$ corresponds to a connected component of the Dynkin diagram for L_Y. So the only possibilities for an A_1 to appear as one of the simple factors of L_Y' are when $\dim(W_i) = 2$ for some $i < l/2$, or $\dim(W_{l/2}) \leq 4$.

To show $\dim(W_i) \geq m$, it suffices to find m T_X-weights of W which occur in W_i. By the result in [12], weights which appear in characteristic 0 also appear in characteristic p. This is the approach we use to obtain contradictions for most embeddings $X \hookrightarrow Y$.

For a T_X-weight ω of W, $\omega + \omega^t = \omega - w_0\omega$ is a sum of roots, and we let l_ω be the height of $\omega + \omega^t$ in the root lattice (the number of summands when we express $\omega + \omega^t$ as a sum of fundamental roots). So for l as above, $l = l_\delta$ where δ is the T_X-high weight of W.

The constructions in this section also apply to the embedding of an arbitrary parabolic subgroup $P_X = Q_X L_X$ of X in a parabolic subgroup P_Y of Y, using Q_X-levels in place of the U_X-levels. The weights appearing in W_i then are those of the form $\delta - e_1\beta_1 - \cdots - e_m\beta_m$ where the sum of the e_j for $\beta_j \in \Pi(X) - \Pi(L_X')$ is i. Again by Lemma 2.4, $W_i \cong [W, Q_X^i]/[W, Q_X^{i+1}]$.

One more fact: If P_X is a t-stable parabolic subgroup of X (including $P_X = B_X$), then each W_i is clearly t-stable (since then $W_{\delta - e_1\beta_1 - e_2\beta_2 - \cdots - e_m\beta_m}$ is sent by t to $W_{\delta - e_m\beta_1 - e_{m-1}\beta_2 - \cdots - e_1\beta_m}$). So P_Y is t-stable. But then $[V, Q_Y]$ is $T_X\langle t\rangle$-stable, hence $[V, Q_Y] = [V, Q_X]$ ($[V, Q_X] \leq [V, Q_Y]$ since $Q_X \leq Q_Y$, and we get equality because $V/[V, Q_X]$ is an irreducible $T_X\langle t\rangle$-module and $T_X \leq L_Y$). So if P_X is t-stable, then $V/[V, Q_Y]$ is a sum of two irreducible L_X'-modules.

The following Lemma provides the basis for the proof of the section's main result; it will also be used throughout the paper.

Lemma 3.2. *If P_Y is a t-stable parabolic subgroup of Y such that $B_X < P_Y$, $U_X < Q_Y$, $T_X < L_Y'$ (where $P_Y = Q_Y L_Y, B_X = U_X T_X$ are the Levi decompositions), then at least one of the simple factors of L_Y' has type A_1; and if this factor corresponds to α_j, then $a_j = 1$. In addition, $a_i = 0$ for $\alpha_i \in \Pi(L_Y')$, $i \neq j$.*

Proof. This is Lemma 5.2 of [4]. □

3.2. **The Cases** $X = A_2$ **and** $X = A_3$**.** We use Lemma 3.2 heavily. As always, $\delta = d_1\delta_1 + d_2\delta_2 + \cdots$ is the T_X-high weight of W. Let β be a nonnegative sum of fundamental roots of X, of height j in the root lattice. Note that if $\delta - \beta$ is a dominant weight such that the X-module with high weight $\delta - \beta$ has $\geq m$ weights at level i, then W has $\geq m$ weights at level $j + i$. So in our attempt to prove that there cannot be a U_X-level of dimension 2 in W, we will proceed by induction on the high weight δ.

$X = A_2$: Since δ is symmetric, $\delta = a\delta_1 + a\delta_2$ for some $a > 0$. Here we will always have $\dim(W_1) = 2$, since the only two weight spaces in level 1 are $\delta - \beta_1$ and $\delta - \beta_2$, both of dimension 1; we will deal with level 1 after we discuss levels 2 and higher. In evaluating the numbers of weights at these levels, we will first use an induction to deal with the case $a \geq 4$, and then deal with $a = 3$ and $a = 2$.

Assume $a = 4$. Then $l = l_\delta = 16$, so we must check to level 8 (we must list three weights at every level 2-7, and 5 at level 8). We have the weights in the table below:

Level	Weights
2	$\delta - 2\beta_1$, $\delta - \beta_1 - \beta_2$, $\delta - 2\beta_2$
3	$\delta - 3\beta_1$, $\delta - 2\beta_1 - \beta_2$, $\delta - 3\beta_2$
4	$\delta - 4\beta_1$, $\delta - 3\beta_1 - \beta_2$, $\delta - 4\beta_2$
5	$\delta - 4\beta_1 - \beta_2$, $\delta - 3\beta_1 - 2\beta_2$, $\delta - 2\beta_1 - 3\beta_2$
6	$\delta - 4\beta_1 - 2\beta_2$, $\delta - 3\beta_1 - 3\beta_2$, $\delta - 2\beta_2 - 4\beta_2$
7	$\delta - 5\beta_1 - \beta_2$, $\delta - 4\beta_1 - 3\beta_2$, $\delta - 3\beta_1 - 4\beta_2$
8	$\delta - 6\beta_1 - 2\beta_2$, $\delta - 5\beta_1 - 3\beta_2$, $\delta - 4\beta_1 - 4\beta_2$, $\delta - 3\beta_1 - 5\beta_2$, $\delta - 2\beta_1 - 6\beta_2$.

So L_Y' has no factors of type A_1 except possibly the factor L_1 corresponding to W_1.

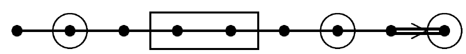
Assume $a > 4$. Then $\delta - \beta_1 - \beta_2 = (a-1)\delta_1 + (a-1)\delta_2$ is dominant, and by induction δ has enough weights at all levels except possibly at levels 1, 2 and 3. At level 2, we have

$\delta - 2\beta_1$, $\delta - \beta_1 - \beta_2$, and $\delta - 2\beta_2$; at level 3, $\delta - 3\beta_1$, $\delta - 2\beta_1 - \beta_2$, and $\delta - 3\beta_2$. So again, L_1 is the only possible A_1-factor of L'_Y.

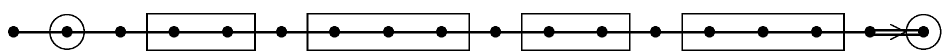
Assume $a = 3$. Then $l = 12$; we must check dimensions to level 6. In levels 2-5, we have enough weights as above. So we must show that W_6 has dimension at least 5. The weights at level 6 are $\delta - 4\beta_1 - 2\beta_2$, $\delta - 3\beta_1 - 3\beta_2$, and $\delta - 2\beta_1 - 4\beta_2$. If $p \neq 7$, then $\dim(W_{\delta - 3\beta_1 - 3\beta_2}) \geq 3$, so $\dim(W_6) \geq 5$. So unless $p = 7$, here again we have only the $L_1 = A_1$ possibility.

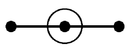
If $a = 2$, then $l = 8$ and we must check dimensions to level 4. For level 2, we have enough weights as above. At level 3, we have $\delta - 2\beta_1 - \beta_2$ and $\delta - \beta_1 - 2\beta_2$; if $p \neq 5$, each has dimension 2, so $\dim(W_3) > 3$. At level 4, the weights are $\delta - 3\beta_1 - \beta_2$, $\delta - 2\beta_1 - 2\beta_2$, and $\delta - \beta_1 - 3\beta_2$. If $p \neq 5$, then $\delta - 2\beta_1 - 2\beta_2$ has dimension ≥ 3, so $\dim(W_4) \geq 5$. As above, unless $p = 5$, we have only the $L_1 = A_1$ possibility.

From the construction of P_Y we can see that in any case covered above (including $(a, p) = (2, 5), (3, 7)$), there is only one node in the Dynkin diagram between the L_i, since there are no U_X-levels of dimension 1 other than $\langle w^+ \rangle$. Also notice that from the $a = 2$ and $a = 3$ cases above we know the embeddings in the cases $(a, p) = (2, 5), (3, 7)$ (we simply compute the dimensions of the levels): If $a = 2, p = 5$, then $\dim(W) = 19$, so Y has type B_9 and P_Y is the parabolic subgroup of Y corresponding to the indicated nodes:

.

If $a = 3, p = 7$ then $\dim(W) = 37$, Y is of type B_{18}, and P_Y corresponds to:

.

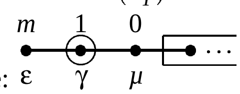
So the possibilities are: 1) $a_2 = 1$, with the first simple factor of L'_Y corresponding to  $\cdots$; 2) $a = 2, p = 5$; 3) $a = 3, p = 7$; 4) $a = 1$. By Lemma 3.2, in the marking for the high weight of V on the Dynkin diagram for Y there is only one non-zero label on the nodes representing L_Y, and this non-zero label must be a 1 on a node corresponding to an A_1 factor of L'_Y; call this node γ. By the comment above, all nodes in the Dynkin diagram are either in or adjacent to $\Pi(L'_Y)$ (except possibly in case 4). Our aim is to show that all nodes except γ have marking 0.

As in the introduction, let $V^2(Q_Y) = [V, Q_Y]/[V, Q_Y, Q_Y]$; similarly define $V^2(U_X)$ and $V_i^2(U_X)$ for $i = 1, 2$. Recall that λ is the T_Y-high weight of V. By Lemma 2.4, the weights in $V^2(Q_Y)$ are those of the form $(\lambda - \beta)|_{T_{L_Y}}$ where $\beta \in \Sigma^+(Y)$ and there exists $\varepsilon \in \Pi(Y) - \Pi(L_Y)$ such that $\beta - \varepsilon \in \Sigma(L_Y)$. Then by Lemma 2.4, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(U_X)) \leq 4$.

First consider the case $a = 1$. If $p \neq 3$, then $\dim(W) = 8$ and $Y = D_4$; if $p = 3$, then $\dim(W) = 7$ and $Y = B_3$. In both cases, constructing P_Y as usual, we have $\alpha_2 \in \Pi(L'_Y)$, so the only possibility for an A_1 factor of L_Y to occur is $\Pi(L'_Y) = \{\alpha_2\}$ (as all other nodes adjoin α_2). For $p \neq 3$, if another node has a non-zero label, say α_3 (all are equivalent by symmetry for this argument), then in $V^2(Q_Y)$ we have the high weights $\lambda - \alpha_3|_{T_{L'_Y}}$, giving a composition factor of dimension 3; and $\lambda - \alpha_2 - \alpha_1$ and $\lambda - \alpha_2 - \alpha_4$ each giving one of dimension 1. But this contradicts $\dim(V^2(Q_Y)) \leq 4$. So $\lambda = \lambda_2$, giving an example of 1) above, which we deal with below. If $p = 3$ and $a_1, a_3 \neq 0$, then $\lambda - \alpha_1|_{T_{L'_Y}}$, $\lambda - \alpha_3|_{T_{L'_Y}}$ each give a composition factor of dimension 3 in $V^2(Q_Y)$, again a contradiction. Finally, irreducible B_3-modules with high weights $e\lambda_1 + \lambda_2$ or $\lambda_2 + e\lambda_3$ are all too large to be the sum of two restricted irreducible A_2-modules unless $e = 0$ (by counting weights that appear in the Weyl module of $\mathcal{W}_{B_3}(e\lambda_1+\lambda_2)$ or $\mathcal{W}_{B_3}(\lambda_2+e\lambda_3)$; all these weights appear in V by [8]). So again $\lambda=\lambda_2$, which we deal with below.

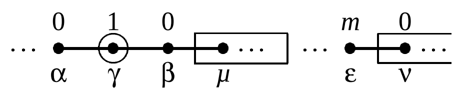
For cases 1)–3), assume there is an $\varepsilon\in\Pi(Y)$ which adjoins γ and has non-zero marking m. If ε is not an end node, then it also adjoins another factor L_l of L_Y' by our comment above that there are never two adjacent nodes outside $\Pi(L_Y')$; then $\lambda-\varepsilon$ is an L_Y'-high weight in $V^2(Q_Y)$, and the L_Y'-high weight module of this high weight has dimension ≥ 6 ($(\lambda-\varepsilon)|_{T_{L_Y'}}=2\lambda_j+\lambda_k$ where λ_j is the fundamental dominant weight corresponding to γ and λ_k is the node of L_l which adjoins ε — the L_Y'-module with this high weight has dimension at least $2\cdot 3=6$). This contradicts $\dim(V^2(Q_Y))\leq 4$.

If ε is an end node, it cannot be the short root in a B_n (that root is in $\Pi(L_Y')$ because we saw that $\dim(W_{l/2})\geq 3$ in all cases). Then we have the picture:

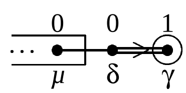


. We have high weights $(\lambda-\varepsilon)|_{T_{L_Y'}}$ of dimension 3 and $(\lambda-\gamma-\mu)|_{T_{L_Y'}}$ of dimension ≥ 2. Again, this contradicts $\dim(V^2(Q_Y))\leq 4$. So all nodes adjoining γ have 0 label.

Assume there is an $\varepsilon\in\Pi(Y)$ which does not adjoin γ and has non-zero marking m. By Lemma 3.2, $\varepsilon\notin\Pi(L_Y')$; and ε adjoins $\Pi(L_Y')$ since every fundamental root is either in $\Pi(L_Y')$ or adjoins it. If γ is not an end node, we have the pictures (different pieces of the Dynkin diagram for Y):



Here $\lambda-\alpha-\gamma|_{T_{L_Y'}}$ is a high weight in $V^2(Q_Y)$, giving dimension ≥ 1; $\lambda-\gamma-\beta|_{T_{L_Y'}}$ gives dimension ≥ 2; and $\lambda-\varepsilon$ gives dimension ≥ 2. Again, this is a contradiction. The node γ can be an end node only in the cases $(a,p)=(2,5),(3,7)$ (these are the only cases in which an A_1 factor of L_Y' corresponds to an end node of the Dynkin diagram for Y), with γ the short root of a B_n; then the picture is:



. Here $\lambda-\gamma-\delta|_{T_{L_Y'}}$ gives dimension ≥ 4; and $\lambda-\varepsilon|_{T_{L_Y'}}$ gives ≥ 1, again a contradiction. So all nodes other than γ must have label 0.

For both of the above non-end node cases, similar arguments hold if there is a double bond in one of the relevant pieces of the Dynkin diagram for Y.

So γ is the only node in the Dynkin diagram with a non-zero label. We need only show that the few remaining possibilities do not lead to examples.

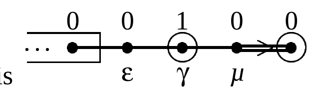
Case 1): V has high weight λ_2. Then $V\cong\bigwedge^2 W$ by [7, II.2.15]. Regard W as an X-module. Remember that $\delta=a\delta_1+a\delta_2$ is the T_X-high weight for W. Let $v_1\in W_\delta$ be a maximal vector in W; $0\neq v_2\in W_{\delta-\beta_1}, 0\neq v_3\in W_{\delta-\beta_2}$. Then $v_1\wedge v_2$ and $v_1\wedge v_3$ are X-maximal vectors in V, so $KX(v_1\wedge v_2)\oplus KX(v_1\wedge v_3)=V$.

We now consider the dimension of $KX(v_1\wedge v_2)$. The vector $v_1\wedge v_2$ has weight $2\delta-\beta_1=(2a-2)\delta_1+(2a+1)\delta_2$. So $\dim(KX(v_1\wedge v_2))\leq\dim(\text{Weyl module})=\frac{1}{2}(2a-1)(2a+2)(4a+1)=8a^3+6a^2-3a-1$. Also, $\dim(KX(v_1\wedge v_3))=\dim(KX(v_1\wedge v_2))$ (since t interchanges them), so $\dim(V)\leq 16a^3+12a^2-6a-2$.

On the other hand, $\dim(W)\geq 3a^2+3a+1$ (this is the number of weights that appear in the Weyl module with the same T_X-high weight as W; all these weights appear in W by Lemma 1.12), and $\dim(V)=\dim(\bigwedge^2 W)=\binom{\dim(W)}{2}\geq\binom{3a^2+3a+1}{2}=(3a^2+3a+1)(3a^2+$

$3a)/2 = (9a^4 + 18a^3 + 12a^2 + 3a)/2$. So $9a^4 + 18a^3 + 12a^2 + 3a \leq 2(16a^3 + 12a^2 - 6a - 2)$. But this has no solutions in positive integers. So this case is ruled out.

The only cases not ruled out by the above are:

Case 2): $(a, p) = (2, 5)$ (here remember $Y = B_9$). If $\lambda = \lambda_7$ then the picture is 

, and $\lambda - \gamma - \varepsilon|_{L_Y}, \lambda - \gamma - \mu|_{L_Y}$ each give dimension ≥ 3. So $\lambda \neq \lambda_7$. If $\lambda = \lambda_9$, then $\dim(V) = 2^9 = 512$. But the dimension of any irreducible A_2-module with high weight $c\delta_1 + d\delta_2$ $(c, d < 5, c \neq d)$ is at most 90. So V is too large to be the sum of two restricted irreducible modules for X.

Case 3): $(a, p) = (3, 7)$ (here $Y = B_{18}$) with $\lambda = \lambda_{18}$. Here $\dim(V) = 2^{18}$. But $c, d < 7, c \neq d$; again, V is too large to be the sum of two restricted irreducible X-modules.

So X is not of type A_2.

$X = A_3$: We use a similar induction. Let $\delta = a\delta_1 + b\delta_2 + a\delta_3$ be the T_X-high weight of W. First we eliminate the case $b = 0$ with the A_1 factor of L'_Y (referred to in Lemma 3.2) corresponding to α_2.

Assume $\delta = a\delta_1 + a\delta_3$ and $\lambda = \lambda_2 + \cdots$. Let $P_X = L_X Q_X$ be the maximal parabolic subgroup of X corresponding to $\beta_3 \in \Pi(X)$, and embed P_X in a parabolic subgroup P_Y of Y via the construction in Lemma 2.7. Now L_1 (the simple factor of L'_Y corresponding to Q_X-level 0 of W) is of type A_2 if $a = 1$, and type A_l with $l \geq 5$ if $a > 1$; and the root system of L_1 contains α_1. (We determine the rank of L_1 by counting weights that appear in Q_X-level 0 of W, using Suprunenko's result [12] that weights which appear in the Weyl module also appear in the irreducible module.)

Assume $V/[V, Q_Y]$ is reducible as an L_X-module; that is,

$$V/[V, Q_Y] =_{L_X} V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X].$$

Then $Z \leq Z(L_Y)$ since $L_Y = C_Y(Z)$. So Z induces scalars on $V/[V, Q_Y]$ (an irreducible L_Y-module).

$$\begin{aligned} Z &= \left\{ \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c^{-3} \end{pmatrix} \mid c \in K^* \right\} \\ &= \{ h_{\beta_1}(c) h_{\beta_2}(c^2) h_{\beta_3}(c^3) \mid c \in K^* \} \end{aligned}$$

Now $h_{\beta_1}(c) h_{\beta_2}(c^2) h_{\beta_3}(c^3)$ acts as multiplication by $c^{b_1 + 2b_2 + 3b_3}$ on a high weight vector $v_1 \in V_1$, and as multiplication by $c^{b_3 + 2b_2 + 3b_1}$ on a high weight vector $v_2 \in V_2$. But by our assumption, v_1 and v_2 both have nonzero images in $V/[V, Q_Y]$; because $b_1 + 2b_2 + 3b_3 \neq b_3 + 2b_2 + 3b_1$, we have a contradiction ($b_1 \neq b_3$ because $V_1 \not\cong_X V_2$).

So $V/[V, Q_Y]$ is an irreducible L_X-module, and thus an irreducible L'_X-module by Lemma 2.2. Assume $V/[V, Q_Y] =_X V_1/[V_1, Q_X]$.

Now we are in the situation studied in [9]: $V/[V, Q_Y]$ is an irreducible module for L'_X (of type A_2) and for L_1. As there are no examples matching this setup in [9, Table 1], we know that either the embedding $L'_X \hookrightarrow L_1$ is an isomorphism, or $V/[V, Q_Y]$ is the natural module for L_1. Both possibilities are excluded if $a > 1$, as then L_1 is of type A_l with $l \geq 5$ (so $L_1 \not\cong L'_X$), and $a_2 = 1$ (so $V/[V, Q_Y]$ is not the natural module for L_1). On the other hand, if $a = 1$ then necessarily $L'_X \cong L_1$ (as $W/[W, Q_Y]$ then has L'_X-high weight δ_1).

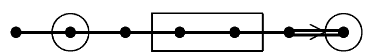
So $V/[V,Q_Y]$ must have L'_X-high weight $a_1\delta_1+\delta_2$, since it has L_1-high weight $a_1\lambda_1+\lambda_2$ and the embedding is an isomorphism. But $V_1/[V_1,Q_X]$ also has high weight $b_1\delta_1+b_2\delta_2$. So $a_1=b_1$ and $b_2=1$.

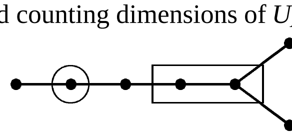
Now the argument above can be repeated with the maximal parabolic subgroup of X corresponding to β_1 instead of β_3, with the conclusion that $b_3=a_1$. But then $b_1=b_3$, a contradiction.

So if $\delta=a\delta_1+a\delta_3$, then $a_2=0$

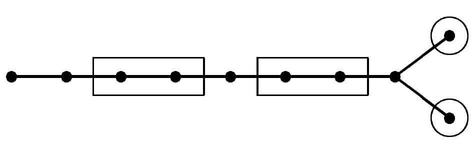
Every weight of the form $a\delta_1+b\delta_2+a\delta_3$ except $\delta_1+\delta_3$, δ_2, and $2\delta_2$ has one of $2\delta_1+2\delta_3$ or $\delta_1+\delta_2+\delta_3$ as a subdominant weight. It is easy to check, as in the A_2 case, that the modules with these latter two high weights have enough weights at every level, so we can proceed by induction: If $b<2$ and $a>2$, then by induction $\delta-\beta_1-\beta_2-\beta_3=(a-1)\delta_1+b\delta_2+(a-1)\delta_3$ has enough weights at all levels; we need to check δ-levels 2-5. As before, there are enough weights in each of these levels, so by induction, L_1 is the only possible factor of L'_Y of type A_1.

If $b\geq 2$, then by induction $\delta-\beta_2$ has enough weights at all levels, and we need check only δ-levels 1 and 2. If $a=0$, then $\dim(W_1)=1$, so L_1 is trivial; if $a>0$, then $\dim(W_1)=3$, so L_1 is of type A_2. Again, there are enough weights at level 2. So there are no possible A_1 factors of L'_Y here.

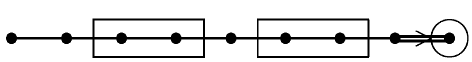
So the possibilities which have not been ruled out are: $\delta=\delta_1+\delta_3$, $\delta=\delta_2$, and $\delta=2\delta_2$. If $\delta=\delta_2$, then $X=SO(W)=Y$, so there are no examples here. If $\delta=\delta_1+\delta_3$, we can check the dimensions of the weight spaces and find that $\dim(W)=15$ if $p\neq 2$, and P_Y is the parabolic subgroup of Y corresponding to the subset $\{\alpha_2,\alpha_4,\alpha_5,\alpha_7\}$ of $\Pi(Y)$, as indicated on this picture of the Dynkin diagram for $Y=B_7$:  .

If $p=2$, then $\dim(W)=14$, and in [9, page 273] it is shown that X stabilizes a quadratic form on W. So in this case $Y=D_7$ and counting dimensions of U_X-levels, we see that the parabolic subgroup P_Y corresponds to  (the stabilizer of the 2-dimensional level 3 is SO_2, which is a torus). So Lemma 3.5 rules out this case.

If $\delta=2\delta_2$, again we can check dimensions of all the weight spaces to find $\dim(W)=20$ $(p\neq 2,3)$; $\dim(W)=19$ $(p=3)$, and P_Y is indicated by the circled nodes of the Dynkin diagram for Y:

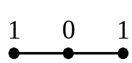
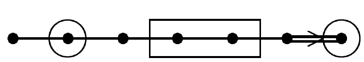
 $(p\neq 2,3)$;

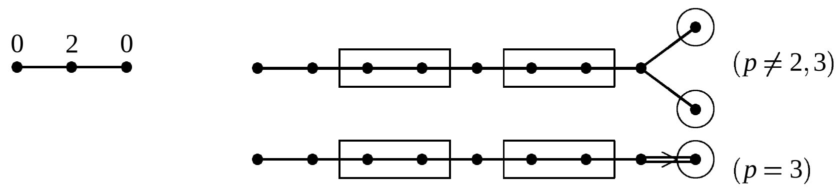
or

 $(p=3)$.

Notice we need not consider $p=2$ by Lemma 2.6.

So the cases we must deal with are:

$W\|_X$	Y and embedding of $B_X<P_Y$
1) 	 $(p\neq 2)$

2) 

In the marking for V on $\Pi(Y)$, there is a 1 on one of the nodes α_j corresponding to an A_1 factor of L'_Y. There can be no other nonzero marking on any of the indicated nodes, since $\dim(V/[V,Q_Y]) = 2$. Recall that $j \neq 2$ by Lemma 3.5; thus, as we see from the pictures above, $j = n$ is the only possibility (or $j = n-1$ in case 2, but we may assume $j = n$ by symmetry). We claim that α_n is the only node with a nonzero label.

Claim 3.3. *In any of the above cases $\lambda = \lambda_n$.*

Proof. Since $U_X \leq Q_Y$, we have $[V,Q_Y,Q_Y] \geq [V,U_X,U_X]$. Then because $[V,Q_Y] = [V,U_X]$, we have $\dim([V,Q_Y]/[V,Q_Y,Q_Y]) \leq \dim([V,U_X]/[V,U_X,U_X]) \leq 6$ by Lemma 2.4. The weights that appear in $V^2(Q_Y) = [V,Q_Y]/[V,Q_Y,Q_Y]$ are those of the form $\lambda - \beta$, where if $\beta = \sum e_i\alpha_i$, then the sum of the e_i for those $\alpha_i \in \Pi(Y) - \Pi(L'_Y)$ is 1. Let us consider the above cases. Remember $\lambda = \sum a_i\lambda_i$ is the T_Y-high weight for V:

1) The node α_7 has label 1 ($a_7 = 1$). Consider the possibilities for another nonzero label: If $a_1 \neq 0$, then $\lambda - \alpha_1$ is a high weight in $V^2(Q_Y)$, giving a composition factor of dimension 4. Another high weight is $\lambda - \alpha_6 - \alpha_7$, giving dimension 6 since $p \neq 2$. But above we noted that $\dim(V^2(Q_Y)) \leq 6$. So $a_1 = 0$. If $a_3 \neq 0$, then $\lambda - \alpha_3$ gives $\dim(V^2(Q_Y)) \geq 12$; so $a_3 = 0$. If $a_6 \neq 0$, then $\lambda - \alpha_6$ gives dimension 12 since $p \neq 2$. So $a_6 = 0$.

2) If $p \neq 3$, we are in the case $Y = D_{10}$, with $a_{10} = 1$. If $a_8 \neq 0$, then $\lambda - \alpha_8$ gives dimension 18 in $V^2(Q_Y)$. If $a_5 \neq 0$, then $\lambda - \alpha_5$ gives dimension 18. If $a_2 \neq 0$, then $\lambda - \alpha_2$ gives dimension 6 and $\lambda - \alpha_8 - \alpha_9$ gives dimension 6. If $a_1 \neq 0$, then $\lambda - \alpha_8 - \alpha_9$ gives dimension 6 and $\lambda - \alpha_1$ dimension 2. So in fact $a_i = 0$ for all $i \neq 9$.

If $p = 3$, then $Y = B_9$ and $a_9 = 1$. If $a_8 \neq 0$, then $\lambda - \alpha_8$ gives dimension 6 and $\lambda - \alpha_8 - \alpha_9$ gives dimension ≥ 1; so $a_8 = 0$. If $a_5 \neq 0$, $\lambda - \alpha_5$ gives dimension 18; if $a_2 \neq 0$, $l - \alpha_2$ gives dimension 6 and $\lambda - \alpha_8 - \alpha_9$ gives dimension 6. If $a_1 \neq 0$, then $\lambda - \alpha_8 - \alpha_9$ gives dimension 6 and $\lambda - \alpha_1$ dimension 2. So $a_i = 0$ for all $i \neq 9$. □

Lemma 3.4. *If $\lambda = \lambda_n$ (i.e. if we are in one of the remaining cases), then as an X-module, W has high weight $2\delta_2$ and V is as in the statement of Theorem 3.1.*

Proof. We still have to check cases 1) and 2) on page 11 with $\lambda = \lambda_n$:

1) Assume W has T_X-high weight $\delta_1 + \delta_3$ and $p \neq 2$. Consider the embedding in a parabolic subgroup $P_Y \leq Y$ of the parabolic subgroup $P_X \leq X$ corresponding to $\{\beta_1, \beta_2\} \subseteq \Pi(X)$. Checking the dimensions at different Q_X-levels as before, we see that for any characteristic, P_Y corresponds to $\{\alpha_1,\alpha_2,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}$. Since $\lambda = \lambda_n, \dim(V/[V,Q_Y]) = 16$ ($V/[V,Q_Y]$ is isomorphic to a spin module for the simple factor of L'_Y of type B_4 corresponding to $\{\alpha_4,\alpha_5,\alpha_6,\alpha_7\}$). The quotient $V/[V,Q_Y]$ is also an irreducible L'_X-module (again by considering the action of $Z = Z(L_X)^\circ$ on the two T_X-high weight vectors of V); but $L'_X = A_2$, which has no irreducible representations of dimension 16 in any characteristic. So we have no examples here.

2) Assume $W|_X$ has high weight $2\delta_2$. If $p = 3$, then $\dim(W) = 19$ and $Y = B_9$. Using P_X corresponding to $\{\beta_1, \beta_2\}$ as above, we get an embedding of P_X in the parabolic subgroup P_Y corresponding to $\Pi(Y) - \{\alpha_6\}$. Since $\lambda = \lambda_n$, $\dim(V/[V,Q_Y]) = \dim(\mathrm{spin}(B_3)) = 8$. But $V/[V,Q_Y]$ is an irreducible L'_X-module (by the action of Z again), and $L'_X = A_2$, which has no 8-dimensional irreducible representations in characteristic 3.

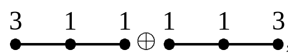
If $p \neq 3$ then take P_X as above; again P_Y corresponds to $\Pi(Y) - \{\alpha_6\}$ and $\dim(V/[V,Q_Y]) = \dim(\mathrm{spin}(D_4)) = 8 = \dim(V_i/[V_i,Q_X])$ for $i = 1$ or 2, say $i = 1$. So $b_1 = 1, b_2 = 1$.

Now let P_X correspond to $\{\beta_1, \beta_3\}$. This P_X is t-stable, so

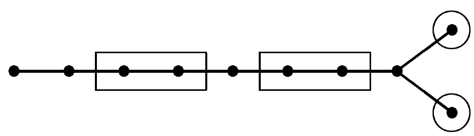
$$V/[V,Q_Y] = V/[V,Q_X] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X].$$

The embedding gives P_Y corresponding to $\Pi(Y) - \{\alpha_1, \alpha_5\}$. So here,

$$\begin{aligned} \dim(V/[V,Q_Y])a = 16 = \dim(V/[V,Q_X]) \\ =& 2\dim(V_1/[V_1,Q_X]) \\ =& 2(b_1+1)(b_3+1) = 4(b_3+1). \end{aligned}$$

So $b_3 = 3$. Now $\dim(V_{D_{10}}(\lambda_n)) = 2^9$ in any characteristic; $\dim(V(3\delta_1+\delta_2+\delta_3)) = 256 = 2^8$ when $p > 7$ or $p = 0$. So $V|_X =$ 

, $V|_Y = \mathrm{spin}(D_{10})$ is a possibility for $p \neq 2,3,5,7$.

In this case, consider again the embedding of B_X in the parabolic subgroup P_Y corresponding to  . We define some temporary notation: Let $\lambda_{1,j} = (\lambda_1 - \alpha_1 - \cdots - \alpha_j)|_{T_X}$. By the construction of the embedding, we know that

$$\begin{aligned} \lambda_1|_{T_X} &= \delta\ (= 2\delta_2) \\ \lambda_{1,1} = (\lambda_1 - \alpha_1)|_{T_X} &= \delta - \beta_2 \\ \{\lambda_{1,2}, \lambda_{1,3}, \lambda_{1,4}\} &= \{\delta - \beta_1 - \beta_2, \delta - \beta_2 - \beta_3, \delta - 2\beta_2\} \\ \{\lambda_{1,5}, \lambda_{1,6}, \lambda_{1,7}\} &= \{\delta - \beta_1 - 2\beta_2, \delta - 2\beta_2 - \beta_3, \delta - \beta_1 - \beta_2 - \beta_3\} \\ \{\lambda_{1,8}, \lambda_{1,9}, \lambda_{1,10}\} &= \{\delta - 2\beta_2 - 2\beta_3, \delta - 2\beta_1 - 2\beta_2, \delta - \beta_1 - 2\beta_2 - \beta_3\}, \end{aligned}$$

with $(\lambda_1 - \alpha_1 - \cdots - \alpha_8)|_{T_X} = -((\lambda_1 - \alpha_1 - \cdots - \alpha_{10})|_{T_X})$ (since $\lambda_1 - \alpha_1 - \cdots - \alpha_8 = -\lambda_8 + \lambda_9 + \lambda_{10} = -(\lambda_1 - \alpha_1 - \cdots - \alpha_{10})$).

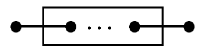
This gives enough information to determine the possibilities for $\alpha_i|_{T_X}$ for $i = 1,\ldots,10$. We can write the T_Y-high weight for V, λ_{10}, in terms of the α_i, and we find that with any of the possible choices made above, $\{\lambda_{10}|_{T_X}, (\lambda_{10} - \alpha_{10})_{T_X}\} = \{3\delta_1 + \delta_2 + \delta_3, \delta_1 + \delta_2 + 3\delta_3\}$. So V contains A_3-submodules of these 2 high weights; since their dimensions add to $\dim(V)$, we have the case stated in the theorem. □

This completes the proof for $X = A_3$.

3.3. **When Lemma 3.2 doesn't help.** Using our standard construction of P_Y (Lemma 2.7), the obvious situation in which the Lemma 3.2 is of no help is when $\delta = a\delta_i + b\delta_j$, i.e. when U_X-level 0 has dimension 2. In this case $L_1 \leq L'_Y$ is of type A_1, corresponding to $\alpha_2 \in \Pi(Y)$. Remember that δ must be symmetric, so that in fact the following is what we will need.

Lemma 3.5. *The situation* $\delta = a\delta_i + a\delta_{m-i+1}$ *(*$i \leq m/2$*),* $a_2 = 1$ *does not give any examples in the Main Theorem if* $m > 3$*.*

Proof. With the given δ and with P_Y as in Lemma 3.2, we have $\alpha_2 \in \Pi(L'_Y)$ since level 1 in the construction of P_Y has dimension 2. So Lemma 3.2 tells us that a_2 is the only nonzero coefficient on $\Pi(L'_Y)$.

Assume $i > 1$. Consider the parabolic subgroup P_X of X corresponding to $\Pi(X) - \{\beta_1, \beta_m\}$, as in the following picture: . So L'_X is of type A_{m-2}, and if we embed $P_X \hookrightarrow P_Y$ by the usual construction, then level 0 (that is, $W/[W,Q_X]$) of the flag which results is totally singular and has dimension ≥ 7 (the smallest $W/[W,Q_X]$ could be is when L'_X is of type A_2 with a label of 1 on each node; this has dimension 7 in characteristic 3 and dimension 8 in other characteristics); thus L_1 (the simple factor of L'_Y corresponding to the quotient $W/[W,Q_X]$ in the flag) is of type A_l for some $l \geq 6$.

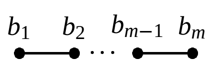
By Lemma 2.11, only one L_j acts nontrivially on $V/[V,Q_Y]$. So $V/[V,Q_Y]$ is a nontrivial (there is at least one nonzero label on L_1 in the marking for the high weight of $V/[V,Q_Y]$, namely, on the second node), irreducible L_1-module, and as an L'_X-module, $V/[V,Q_Y]$ is the sum of two irreducibles, $V_1/[V_1,Q_X]$ and $V_2/[V_2,Q_X]$. The natural module for L_1 is isomorphic to $W/[W,Q_Y]$ and is irreducible as an L'_X-module. So unless $V_1/[V_1,Q_X] \cong V_2/[V_2,Q_X]$, we are in the situation we consider in this section and by induction no examples arise (the single case which arises below for $X = A_3$ does not arise inductively because δ does not have the form we are considering here).

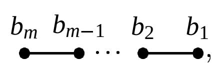
Let the T_X-high weight of V_1 be $b_1\delta_1 + b_2\delta_2 + \cdots + b_m\delta_m$ (so the high weight of V_2 is $b_m\delta_1 + b_{m-1}\delta_2 + \ldots$). Assume $b_2 = b_{m-1}, b_3 = b_{m-2}, \ldots$ (i.e. $V_1/[V_1,Q_X] \cong V_2/[V_2,Q_X]$). This implies $b_1 \neq b_m$ since $V_1 \ncong V_2$. Now take P_X to be another parabolic subgroup of X, corresponding to $\Pi(X) - \{\beta_m\}$, and embed $P_X \hookrightarrow P_Y$ via the same construction. Again we have $W/[W,Q_Y] = W/[W,Q_X]$ irreducible for L'_X and for L_1.

We show that in this case $V/[V,Q_Y]$ is irreducible as an L_X-module (in contrast to the situation when the parabolic subgroup of X is t-stable, which forces $V/[V,Q_Y]$ to be the sum of two irreducibles for L_X): Let $Z = Z(L_X)^\circ$. Since $L_Y = C_Y(Z)$, we have $Z \leq Z(L_Y)$, so Z induces scalars on $V/[V,Q_Y]$ (since L_Y acts irreducibly on this module). But if $V/[V,Q_Y]$ is reducible as an L_X-module, then $V/[V,Q_Y] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X]$, and we show that Z acts differently on these two L_X-modules:

$$\begin{aligned} Z &= \{\mathrm{diag}(a,\ldots,a,a^{-m}) | a \in K^*\} \\ &= \{h_{\beta_1}(a)h_{\beta_2}(a^2)\ldots h_{\beta_m}(a^m) | a \in K^*\}. \end{aligned}$$

The two X-modules V_1, V_2 have high weight labelling



and


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so $h_{\beta_1}(a)h_{\beta_2}(a^2)\ldots h_{\beta_m}(a^m)$ acts as multiplication by $a^{b_1+2b_2+\cdots+mb_m}$ on a high weight vector $v_1 \in V_1$ and as multiplication by $a^{b_m+2b_{m-1}+\cdots+mb_1}$ on a high weight vector $v_2 \in V_2$; these two exponents are not equal. Since v_j has a nonzero image in $V_j/[V_j,Q_X]$, this shows that only one of the $V_j/[V_j,Q_Y]$ can have a nonzero image in $V/[V,Q_Y]$. So $V/[V,Q_Y]$ is irreducible as an L_X-module.

Assume V_1 is the summand which projects nontrivially to $V/[V,Q_Y]$ (so $V_2 \subseteq [V,Q_Y]$). As no irreducible restricted representations of A_{m-1} are tensor decomposable by Lemma 2.1, L_1 is the only simple factor of L'_Y to act nontrivially on $V/[V,Q_Y]$. Note that the rank of L_1 is ≥ 15, since this rank is one less than the dimension of $W/[W,Q_Y]$: With δ of the form we are assuming, the A_{m-1}-high weight of $W/[W,Q_Y]$ is of the form $(a\delta_i + a\delta_{m-i+1})|_{T_{L'_X}}$. The Weyl module for A_{m-1} with this high weight has at least 16 weights, and these weights all appear in the irreducible module $W/[W,Q_Y]$ by the result in [12].

So now we are inductively in the situation examined in [9]: $V/[V,Q_Y]$ is an irreducible module for both L'_X and L_1, and it is not the natural module for L_1 (since in the labelling for the L_1-high weight of $V/[V,Q_Y]$, there is a 1 on the second node of $\Pi(L_1)$). Also, $L'_X \not\cong L_1$ because the natural module for L_1, $W/[W,Q_Y]$, has high weight $\delta|_{T_{L'_X}}$ and thus has dimension larger than m, which is the dimension of the natural module for L'_X. So any examples here would appear in Table 1 of [9]; examining that table, we see that in fact there are no examples. This completes the case $i > 1$.

So we need to consider only the case $i = 1$, i.e. $\delta = a\delta_1 + a\delta_m$, with $\lambda = \lambda_2 + \cdots$. Let $P_X = L_X Q_X$ be the maximal parabolic subgroup of X corresponding to $\beta_j \in \Pi(X)$ and embed P_X in a parabolic subgroup $P_Y = L_Y Q_Y$ of Y via the usual construction. Notice that L_1 (the simple factor of L'_Y corresponding to the Q_X-level 0 of W) is of type A_l, with $l > 3$ unless $m = 4$ (we have taken care of the cases $m = 2,3$ in 3.2). We wish to show that for at least one choice of j, $V/[V,Q_Y]$ is irreducible as an L_X-module. We will again use the action of $Z = Z(L_X)^\circ$ on T_X-high weight vectors in V.

$$\begin{aligned} Z &= \{\mathrm{diag}(\underbrace{a^{(m-j+1)}, a^{(m-j+1)}, \ldots, a^{(m-j+1)}}_{j}, \underbrace{a^{-j}, a^{-j}, \ldots, a^{-j}}_{m-j+1}) \mid a \in K^*\} \\ &= \{h_{\beta_1}(a^{(m-j+1)}) h_{\beta_2}(a^{2(m-j+1)}) \ldots \\ &\qquad h_{\beta_j}(a^{j(m-j+1)}) h_{\beta_{j+1}}(a^{j(m-j)}) h_{\beta_{j+2}}(a^{j(m-j-1)}) \ldots h_{\beta_m}(a^j) \mid a \in K^*\} \end{aligned}$$

If $V/[V,Q_Y]$ is reducible as an L_X-module, then as above, Z must act as multiplication by the same scalar on a high weight vector $v_1 \in V_1$ as on a high weight vector $v_2 \in V_2$, and we get the equation:

$$\begin{aligned} (m-j+1)b_1 + {} & 2(m-j+1)b_2 + \cdots + j(m-j+1)b_j + \\ & j(m-j)b_{j+1} + j(m-j-1)b_{j+2} + \cdots + j \cdot 1 \cdot b_m \\ & = (m-j+1)b_m + 2(m-j+1)b_{m-1} + \cdots + j(m-j+1)b_{m-j+1} + \\ & \qquad j(m-j)b_{m-j} + j(m-j-1)b_{m-j-1} + \cdots + j \cdot 1 \cdot b_1. \end{aligned} \tag{1}$$

If we assume that $V/[V,Q_Y]$ is reducible as an L_X-module for every j, then we have a system of equations which together imply $b_1 = b_m, b_2 = b_{m-1}, \ldots$. For example, the equations for $j = m$ and $j = m-1$ are:

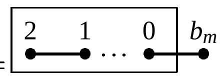
$$\begin{aligned} b_1 + 2b_2 + \cdots + (m-1)b_{m-1} + mb_m &= b_m + 2b_{m-1} + \cdots + mb_1 \\ 2b_1 + 4b_2 + \cdots + 2(m-1)b_{m-1} + (m-1)b_m &= 2b_m + 4b_{m-1} + \cdots + (m-1)b_1. \end{aligned}$$

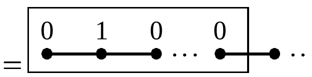
Twice the first equation minus the second gives $(m+1)b_m = (m+1)b_1$. Knowing $b_1 = b_m$, the equations for $j = m$ and $j = m-2$ give $b_2 = b_{m-1}$; continuing in this manner we obtain $b_l = b_{m-l+1}$ for every l. But this is impossible, as it would imply that V is reducible for $X\langle t\rangle$. So for some $j, 1 \le j \le m$, $V/[V,Q_Y]$ is irreducible as an L'_X-module, where L_X, Q_Y are as above.

But then again we are in the situation examined in [9]: $V/[V,Q_Y]$ is an irreducible module for $L'_X = A_{j-1} \times A_{m-j}$ and for L_1. So one of the following occurs:

1. The embedding $L_X \hookrightarrow L_1$ is an isomorphism.
2. $V/[V,Q_Y]$ is the natural module for L_1 (which happens only for $m = 3,4$, since L_1 has rank ≥ 4 in other cases we consider and the L_1-high weight of $V/[V,Q_Y]$ has a nonzero label on the node α_2).
3. $V/[V,Q_Y]$ appears in Table 1 of [9].

We deal with 3 first. Of the appearances of the inclusion $A_{j-1} \times A_{m-j} \le A_l$ in Table 1 of [9], only one (case I_7 there) gives the correct restriction of the natural module for A_l to the subgroup. So the possible picture here is ($L'_X = A_{m-1}$; $L_1 = A_{(m^2+m-2)/2}$; the high weight of $W|_X = 2\delta_1 + 2\delta_m$):

$V_1/[V_1, Q_X] =$ 

and $V/[V, Q_Y] =$ 

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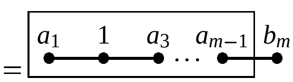
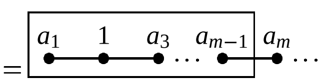
Now we look at yet another parabolic subgroup of X: Let P_X be generated by B_X and the root subgroups for β_1 and β_m; embed P_X in a parabolic subgroup P_Y of Y as usual. By Lemma 2.11, L_1 is the only simple factor of L'_Y to act nontrivially on $V/[V, Q_Y]$. Notice $\dim(W/[W, Q_X]) = 9$ (since $W|_X$ has T_X-high weight $2\delta_1 + 2\delta_m$), so L_1 has rank 8. Thus $2 \cdot 3(b_m + 1) = \dim(V_1/[V_1, Q_X]) + \dim(V_2/[V_2, Q_X]) = \dim(V/[V, Q_X]) = \dim(V/[V, Q_Y]) = \binom{9}{2} = 36$, which tells us $b_m = 5$. This tells us that the only two $T_{L'_X}$-high weights of $V/[V, Q_Y]$ are $(2\delta_1, 5\delta_m)$ and $(5\delta_1, 2\delta_m)$. But in fact $V/[V, Q_Y] \cong \bigwedge^2(W/[W, Q_Y])$ ($p \neq 2$ because of the 2 appearing in the picture above and our assumption that the T_X-high weights of V are restricted), and two $T_{L'_X}$-high weights of $\bigwedge^2(W/[W, Q_Y])$ are $(4\delta_1, 2\delta_m)$ and $(2\delta_1, 4\delta_m)$. Again we have a contradiction.

Next consider item 2. This can only occur if $\dim(W/[W, Q_X]) \leq 4$ (since there is a 1 on the second node in the marking for the high weight of $V/[V, Q_Y]$ on L_1); and this occurs only in the cases we excluded ($X = A_2, A_3$) and the case $X = A_4$, with $j = 4$ (or, equivalently, $j = 1$), $\delta = \delta_1 + \delta_4$. But this in fact gives an instance of item 1 (it implies that $W/[W, Q_Y]$ has $T_{L'_X}$-high weight δ_1).

So we are left with item 1. Note that the equations (1) hold for all $j \neq 1, m$, since in the consideration of item 3 above we obtained a contradiction to $V/[V, Q_Y]$ being irreducible for L'_X when $j \neq 1, m$. We have the pictures

$V_1/[V_1, Q_X] =$ 

$V/[V, Q_Y] =$ 

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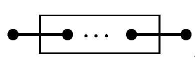
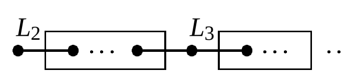
So $a_1 = b_1, 1 = a_2 = b_2, \ldots, a_{m-1} = b_{m-1}$. Now let P_X correspond to the indicated nodes: 

, and embed it in a parabolic subgroup of Y as usual. Then $V/[V, Q_Y]$ is the sum of two irreducible L'_X-modules, and the simple factor L_2 of L'_Y corresponds to the indicated nodes at the beginning of the Dynkin diagram for Y: 

, with L_2 of rank $2m - 3$. But by the construction of P_Y, we know the embedding $L'_X \hookrightarrow L_2$ here (the natural module for L_2 is Q_X-level 1 of W and restricts to L'_X as the sum of the two irreducible modules with high weights $\delta_2|_{T_{L'_X}}$ and $\delta_{m-1}|_{T_{L'_X}}$). As this situation gave no examples in [4], we know by induction that either: 1) $a_3 = a_4 = \cdots = a_{2m-2} = 0$ (i.e. $V/[V, Q_Y] \cong$ the natural module for L_2); or 2) $1 = a_2 = a_{m-1}, a_3 = a_{m-2}, \ldots$ (i.e. $V/[V, Q_Y]$ is reducible for $L'_X\langle t \rangle$).

We noted that all the equations (1) on page 15 hold for $j \neq 1, m$. The equation for $j = m - 1$ is:

$$2b_1 + 4b_2 + \cdots + 2(m-1)b_{m-1} + (m-1)b_m = 2b_m + 4b_{m-1} + \cdots + (m-1)b_1.$$

Now if 2) above holds, we know that $b_2 = b_{m-1}, b_3 = b_{m-2}, \ldots$ (remember that $a_i = b_i$ for $1 < i < m$), which together with the above equation gives $b_1 = b_m$, which is a contradiction. So 2) does not occur.

If $a_3 = \cdots = a_{2m-2} = 0$, i.e. 1) holds, then once again we examine the equation above and see that now it reduces to $2b_1 + 4 + (m-1)b_m = 2b_m + 2(m-1) + (m-1)b_1$ (remembering $a_i = b_i$ for $1 < i < m$ and $a_2 = b_2 = 1$), or $b_m - b_1 = 2$. Let P_X be the parabolic subgroup of X corresponding to $\{\beta_1, \beta_m\}$. Then L_1 is of type A_3, L_1 is the only factor of L_Y' acting nontrivially on $V/[V,Q_Y]$ by Lemma 2.11, and the picture is:

$$V_1/[V_1,Q_X] = \overset{b_1}{\bullet} \otimes \overset{b_1+2}{\bullet} \quad \text{and} \quad V/[V,Q_Y] = \boxed{\overset{b_1}{\bullet}\!\!-\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!-\!\!\overset{0}{\bullet}}\cdots \ .$$

By the Andersen-Jantzen sum formula ([1]), the A_3-module with the high weight pictured above is in fact the Weyl module except when $b_1 = p-2$. But $p > b_m = b_1 + 2$ here, so the dimension of $V/[V,Q_Y]$ is the dimension of the Weyl module, which is $(b_1+1)(b_1+3)(b_1+4)/2$. On the other hand, $\dim(V/[V,Q_Y]) = \dim(V/[V,Q_X]) = 2\dim(V_1/[V_1,Q_X]) = 2(b_1+1)(b_1+3)$. These two equations together imply $b_1 = 0$.

So now all the b_i are known: $b_2 = 1$, $b_m = 2$, and all others are 0. Since one of the coefficients is greater than 1, we know $p \neq 2$. And since the T_X-high weight of W is $\delta_1 + \delta_m$, we know $\dim(W) = m(m+2)$ or $m(m+2)-1$. We claim that $\dim(V) \geq \dim(\bigwedge^2 W)$ (this is clear if $\lambda = \lambda_2$, as then $V = \bigwedge^2 W$ since $Y = D_n$ and $p \neq 2$).

The T_Y-high weight λ of V has a λ_2-coefficient of 1, and W is orthogonal. We claim that any B_n or D_n weight of this form has one of λ_2, λ_3, or λ_n as a subdominant weight. For B_n, every fundamental weight except λ_n is a sum of roots, and $2\lambda_n$ is a sum of roots, so any λ with a λ_2-coefficient of 1 has λ_2 or λ_n as a subdominant weight. For D_n, any fundamental weight λ_k for even $k \leq n-2$ is a sum of roots; for odd k with $1 < k \leq n-2$, λ_k differs from a sum of roots by λ_3. The weight $2\lambda_n$ or $2\lambda_{n-1}$ is either a sum of roots or differs from one by λ_3; finally, $\lambda_1 + \lambda_2$ has λ_3 as a subdominant weight. So the claim holds.

Say $\lambda \succ \lambda_i$. Then by the result in [8], every weight which appears in the Weyl module with T_Y-high weight λ appears in V. So

$$\begin{aligned}\dim(V) \geq\, &\mathrm{card}(\{\omega \text{ is a weight occurring in } V_Y(\lambda_i)\}) \\ &+ \mathrm{card}(\text{Weyl group-orbit of } \lambda) \\ \geq\, &(\binom{\dim(W)}{2} - \dim(W)) + \dim(W) = \dim(\bigwedge^2 W).\end{aligned}$$

Now we have a chain of inequalities (the second line is a computation of the dimension of the Weyl module with the specified high weight, using the Weyl dimension formula):

$$\begin{aligned}\dim(V) &= 2\dim(V_{A_m}(\lambda_2 + 2\lambda_m)) \\ &\leq 2((m-1)m(m+2)(m+3)/4) \\ &= (m^4 + 4m^3 + m^2 - 6m)/2 \\ &< (m^4 + 4m^3 + m^2 - 6m + 2)/2 \\ &= \binom{m(m+2)-1}{2} \\ &\leq \binom{\dim(W)}{2} \\ &\leq \dim(V),\end{aligned}$$

which is a contradiction. So we have ruled out all possible configurations, and the proof of the Lemma is complete. □

3.4. **The General $X = A_m$ Case.** We must prove that there are no triples (X,Y,V) with X acting irreducibly on W and t acting on W, for X of type A_m with $m > 3$. We use the same argument as in the A_2 and A_3 cases to limit the possibilities for the embeddings $X \hookrightarrow Y$, relying on Lemma 3.2. Lemma 3.5 tells us we need not worry about level 1 in the computation of dimensions of U_X-levels.

As in the small cases, the method we use to generate weights that appear in a representation is simple: If μ is a weight in the X-module M and $\langle \mu, \beta_i \rangle \geq a > 0$, then $\mu - a\beta_i$ is another weight appearing in M.

Every symmetric weight for $X = A_m$ except $\delta_1 + \delta_m$ has either $\delta_2 + \delta_{m-1}$ or $\delta_{(m+1)/2}$ as a subdominant weight ($\delta_i + \delta_{m-i+1}$ is a sum of roots for any $1 \leq i \leq m$). It is relatively simple to find enough weights in each level for $\delta = \delta_2 + \delta_{m-1}$: The level l_δ of the low weight $-\delta$ in this case is $2(m + (m-2)) = 4(m-1)$, so we must show there are at least three weights at level j for $2 \leq j < 2(m-1)$, and at least 5 at level $2(m-1)$; this is easy. Unless otherwise stated, the levels we discuss are U_X-levels.

The case $\delta = \delta_{(m+1)/2}$ (m odd) is considerably more difficult. At U_X-level 2 there are only the two weights $\delta - \beta_{(m+1)/2} - \beta_{(m-1)/2}$ and $\delta - \beta_{(m+1)/2} - \beta_{(m+3)/2}$; so Lemma 3.2 is of no help here. We have $l_\delta = (m+1)^2/4$, so if we can show that there are at least 3 weights at each level $3 \leq j < (m+1)^2/8$, and at least 5 at level $(m+1)^2/8$ (when this is integral), we will only have level 2 to worry about.

Notice that for any weight δ, the dimensions and numbers of weights of U_X-levels of the X-module with high weight δ are symmetric about level $l_\delta/2$. In other words, $\dim(W_i) = \dim(W_{l_\delta - i})$ and the same numbers of weights appear in these two spaces, since w_0 interchanges them. So, for instance, if $V_{A_l}(\delta)$ has at least 3 weights at all levels j for $i \leq j \leq l_\delta/2$, then it has at least 3 at all levels j for $i \leq j \leq l_\delta - i$.

For $m = 5, 7$, it is easy to see that there are enough weights at levels 3 through $l_\delta/2$ (three at every level except level 8 for $m = 7$, in which case there are at least 5 weights). We proceed by induction on m (considering the subsystem group of X of type A_{m-2}, corresponding to $\Pi(X) - \{\beta_1, \beta_m\}$): Assume $\delta = \delta_{(m+1)/2}$ for $m \geq 9$. Notice that $\delta_1 - \delta_{(m+1)/2} + \delta_m$ is a weight at level $l_{(\delta_{(m+1)/2}|_{A_{m-2}})} = (m-1)^2/4$, and that if $\delta_{(m+1)/2} - \sum_{i=2}^{m-1} c_i\beta_i$ is a weight at level j in the A_{m-2}-module with high weight $\delta_{(m+1)/2}$, then $\delta_{(m+1)/2} - \sum_{i=2}^{m-1} c_i\beta_i$ is a weight at level j in W for $\delta = \delta_{(m+1)/2}$. So, using the comment in the last paragraph and by induction from the A_{n-2} case, W has at least three weights at levels 3 through $((m-1)^2/4) - 2$. But $((m-1)^2/4) - 2 > (m+1)^2/8$ for $m > 7$, which shows that the only possibilities for a simple factor of L'_Y of type A_1 are the above-mentioned factor corresponding to U_X-level 2, and level $(m+1)^2/8$ when this number is integral.

We need to show that there are at least 5 weights at level $(m+1)^2/8$ when $4 | m+1$ and $\delta = \delta_{(m+1)/2}$. It is easy to write down 5 such weights for $m = 7$, as noted above; in particular in this case there are two at this level which are symmetric (with respect to the graph automorphism). So assume $m \geq 11$ and consider the A_{m-4}-subsystem subgroup of X, corresponding to $\Pi(X) - \{\beta_1, \beta_2, \beta_{m-1}, \beta_m\}$. Assume there are at least two symmetric weights at level $(m-3)^2/8$ for the A_{m-4}-module with high weight $\delta_{(m+1)/2}$; as in the last paragraph each corresponds to a weight of W at level $(m-3)^2/8$. Let γ be one of these weights of W. Write $\gamma = \delta - \sum_{i=3}^{m-2} c_i\beta_i$. For any symmetric weight of W expressed in this form, either:

1) $c_{(m+1)/2} = c_{(m-1)/2} + 1$ $(= c_{(m+3)/2} + 1)$; or

2) $c_{(m+1)/2} = c_{(m-1)/2}$ $(= c_{(m+3)/2})$.

If γ satisfies 1), then $\gamma - \beta_2 - \beta_3 - \cdots - \beta_{(m-1)/2} - 2\beta_{(m+1)/2} - \beta_{(m+3)/2} - \cdots - \beta_{m-1}$ is a symmetric weight of W at level $(m-3)^2/8 + m - 1 = (m+1)^2/8$ which satisfies 2); if γ satisfies 2), then $\gamma - \beta_1 - \beta_2 - \cdots - \beta_{(m-1)/2} - \beta_{(m+3)/2} - \cdots - \beta_m$ is such a weight which satisfies 1). Since of the two symmetric weights at level 8 for $m = 7$, one $(\delta_4 - \beta_1 - \beta_2 - \beta_3 - 2\beta_4 - \beta_5 - \beta_6 - \beta_7)$ satisfies 1) and the other $(\delta_4 - \beta_2 - 2\beta_3 - 2\beta_4 - 2\beta_5 - \beta_6)$ satisfies 2), we conclude that the two symmetric weights at level $(m+1)^2/8$ we obtain are in fact distinct. At the same level we have the weights $\gamma - \beta_1 - \cdots - \beta_{m-1}$ and $\gamma - \beta_2 - \cdots - \beta_m$. So from each of the two symmetric weights at level $(m-3)^2/8$, we obtain 3 weights at level $(m+1)^2/8$; the six weights occurring are all distinct. So there are no possible A_1 factors of L_Y' here.

Note that $\delta = \delta_{(m+1)/2}$ is the only possible high weight of W which does not have at least 3 weights at level 2. Now the induction is much the same as for $X = A_3$: Recall that $\delta = \sum d_i\delta_i$. If $d_{(m+1)/2} > 1$, then $\delta - \beta_{(m+1)/2}$ is lower in the partial order and still has one of $\delta_2 + \delta_{m-1}$, $\delta_{(m+1)/2}$ as a subdominant weight, so by induction δ has enough weights at levels 4 through $(l_\delta/2) - 1$, and we can easily check levels 1, 2, and 3.

If $d_{(m+1)/2} \leq 1$ or m is even, then let k be such that $d_k \neq 0$ but $d_i = 0$ for all $k < i < (m+1)/2$. Then $\delta - \beta_k - \beta_{k+1} - \cdots - \beta_{m-k+1}$ (at level $m - 2k + 2$) is lower in the partial order and still has one of $\delta_2 + \delta_{m-1}$, $\delta_{(m+1)/2}$ as a subdominant weight, so by induction δ has enough weights at levels $m - 2k + 4$ and higher. Once again, the missing levels are easy to check.

So the cases not ruled out by the induction are: $\delta = \delta_1 + \delta_m$, and $\delta = \delta_{(m+1)/2}$ at level 2. For the first, we can easily write down enough weights in levels 2 through $m - 1 = (l_\delta/2) - 1$. At level m the only weight is 0, but the dimension of its weight space is m or (if $p|m+1$) $m-1$. So the only possibilities for a U_X-level of dimension 4 or less are $m = 4$, or $m = 5$ with $p = 2$ or 3.

Consider the case $\delta = \delta_{(m+1)/2}$ $(m \geq 5)$. In the labelling for the Y-high weight of V, there is a 1 on the third node of the Dynkin diagram. Let P_X correspond to $\Pi(X) - \{\beta_1, \beta_m\}$. Then $\dim(W/[W, Q_X]) \geq 6$, so the corresponding simple factor L_1 of L_Y' is of type A_l for some $l \geq 5$, and $V/[V, Q_Y]$ is an irreducible L_1-module whose high weight has a λ_3-coefficient of 1 (L_1 is the only simple factor of L_Y' to act on $V/[V, Q_Y]$ by Lemma 2.11). The natural module for L_1 is isomorphic to $W/[W, Q_Y]$ and is irreducible as an L_X'-module, while $V/[V, Q_Y]$ is the sum of two irreducible modules for L_X', interchanged by t. Since $V/[V, Q_Y] \ncong$ the natural module for L_1 and no cases with this configuration appeared for $X = A_3$, by induction we know that $V/[V, Q_Y]$ must not be irreducible for $L_X'\langle t\rangle$. So $b_2 = b_{m-1}, b_3 = b_{m-2}$, etc. $(V_1|_X = V_{A_m}(b_1\delta_1 + \cdots + b_m\delta_m))$.

Now let P_X correspond to $\Pi(X) - \{\beta_m\}$. By the same construction (using the action of Z on high weight vectors) as on page 14, $V/[V, Q_Y]$ is irreducible as an L_X'-module. The rank of L_1 is at least 9 (since $W/[W, Q_Y] \cong$ the A_{m-1}-module with high weight $\lambda_{(m+1)/2}$), and $V/[V, Q_Y]$ is irreducible as an L_1-module, with a 1 on the third node of the Dynkin diagram in its high weight labelling (L_1 is the only factor of L_Y' acting nontrivially on $V/[V, Q_Y]$ since no restricted irreducible A_{m-1}-modules are tensor decomposable). So we are again in the case examined in [9]. Examining Table 1 there, we see that there are no examples of the configuration we obtain. So $\delta = \delta_{(m+1)/2}$ does not occur.

The two cases that remain are: 1) $m = 4$, $\delta = \delta_1 + \delta_4$, with the T_Y-high weight λ of V having a λ_n-coefficient of 1 (Y has type D_{12} ($p \neq 2,5$), B_{11} ($p = 5$), or possibly C_{12} ($p = 2$)); and 2) $m = 5$, $p = 5$, $\delta = \delta_1 + \delta_5$, λ with a λ_{17}-coefficient of 1 (Y has type D_{17}). In both these cases, we can use the fact that $\dim(V^2(Q_Y)) \leq m \dim(V^1(Q_Y))$ (Lemma 2.9) to conclude that in fact $\lambda = \lambda_n$; i.e. V is a spin module for Y.

For 1), we first let P_X correspond to $\{\beta_2, \beta_3\} \subseteq \Pi(X)$, and conclude that $b_2 = b_3$ as the resulting configuration does not appear in [4]. Next let P_X correspond to $\Pi(X) - \{\beta_4\}$. Then again the construction on page 14 using the action of Z tells us that $V/[V, Q_Y]$ is irreducible as an L'_X-module, and we are back in the situation examined in [9]. Examining Table 1 of [9], we see that there are no examples of modules irreducible for both $A_3 \leq D_8$.

For 2), we use the same constructions and conclude that to have an example, there must be an example of the form $A_4 \leq D_{12}$ in Table 1 of [9]; there is none. So this final possibility is ruled out.

This completes the proof of the theorem. □

4. THE CASE $X = D_m$

In this section we establish the main result for the case $X = D_m$, $G = X\langle t\rangle$. Section 3 included the base case of $X = D_3 = A_3$. So throughout this section $X = D_m$ for $m > 3$ and $G = X\langle t\rangle$. We assume that t acts on W, the natural module for Y.

All notation ($X \leq Y$, V, V_1, V_2, λ, δ, etc.) is as in previous sections. Recall that $\Pi(X) = \{\beta_i\}$ is the set of simple roots of X; $\Pi(Y) = \{\alpha_i\}$ is the set of simple roots of Y; and n is the rank of Y. The main theorem of this section is:

Theorem 4.1. *If X acts irreducibly on the natural module W for Y and X is of type D_m for $m > 3$, then $p = 2$, $Y = C_m$ or B_m, W is the natural module for X and Y; and V is the spin module for Y, the sum of two spin modules for X.*

Proof. Assume X is of type D_m. Let δ be the X-high weight of W. Since we are assuming t acts on W, we have that δ is symmetric with respect to t, so by [9, 1.8], X stabilizes a non-degenerate bilinear form on W. So Y is of type B_n, C_n, or D_n.

Let P_X be the maximal parabolic subgroup of X corresponding to β_1. Then L'_X is of type D_{n-1}, and we embed P_X in P_Y, a parabolic subgroup of Y, via the construction detailed in Lemma 2.7. Then $Q_X \subseteq Q_Y$ and $L'_X \subseteq L'_Y$ for $L_Y = C_Y(Z(L_X)^\circ)$. By Lemma 2.8, L_Y is t-stable. Write $L'_Y = L_1 \times \cdots \times L_m$ where each L_i is simple. By Lemma 2.2, $V/[V, Q_Y]$ is irreducible as an L'_Y-module with high weight $\lambda|_{T_Y \cap L'_Y}$; also, Lemma 2.11 tells us that only one L_i acts non-trivially on $V/[V, Q_Y]$. Note that since P_X is t-stable, Lemma 2.8 tells us that $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$ is a sum of two irreducible L'_X-modules; neither of these L'_X summands is trivial, since that would imply that $V_1 \cong V_2$.

Knowing this, we can list the possibilities for $L_i, (V/[V, Q_Y])|_{L_i}, (V/[V, Q_Y])|_{L'_X}$ based on our inductive knowledge about L'_X (which is of type D_{m-1}):

1. $V/[V, Q_Y] \cong$ the natural module for L_i.
2. (U_3 in Table 2) The natural module W_i for L_i is reducible as an L'_X-module and different from $V^1(Q_Y) = V/[V, Q_Y]$, with L_i of type D_m, $(V/[V, Q_Y])|_{L_i} = \text{spin}(L_i)$, and $(V/[V, Q_Y])|_{L'_X} = \text{spin}(L'_X) \oplus \text{spin}(L'_X)$ (one of the two cases in which $X = D_m$ occurs in the situation examined in [4]).
3. The other case from [4] (U_2 in Table 2): Same as 2 above, except here L_i is of type B_{m-1} and $(V/[V, Q_Y])|_{L_i}, (V/[V, Q_Y])|_{L'_X}$ are as in the statement of [4, Theorem 3.3].
4. W_i is irreducible for L'_X.

Here is a lemma to simplify things considerably:

Lemma 4.2. *In all of the four cases above, if $\alpha \in \Pi(Y)$ and $\langle \alpha, \Pi(L_i) \rangle = 0$, then $\langle \lambda, \alpha \rangle = 0$.*

Proof. Nodes in L_j for $j \neq i$ must have 0 label since L_j acts trivially on $V^1(Q_Y)$. Assume $\langle \lambda, \alpha \rangle \neq 0$ for some $\alpha \in \Pi(Y)$ with $\langle \alpha, \Pi(L_Y) \rangle = 0$ (i.e. α does not adjoin $\Pi(L_Y)$). Define K_Y^α as in the discussion following 2.9. Then Q_Y/K_Y^α has dimension 1 by [9, 3.1] (indeed, $Q_Y/K_Y^\alpha \cong U_{-\alpha}$). It has an L_X-submodule $Q_X K_Y^\alpha / K_Y^\alpha \cong Q_X/(Q_X \cap K_Y^\alpha)$ by [9, 3.3]. By Lemma 2.10(i), $Q'_X = K^{\beta_1} = \langle U_\beta \mid \beta \in \Sigma^-(X), \beta_1\text{-coefficient of } \beta < -1 \rangle = 1$ (since D_m has no roots with a β_1-coefficient less than -1). Then by part (ii) of the same Lemma, $Q_X = Q_X/K^{\beta_1} = Q^{\beta_1}$ is an irreducible L_X-module of high weight $-\beta_1|_{T_{L_X}}$ (and thus dimension $2(m-1)$). But $Q_X \cap K_Y^\alpha$ is a submodule since K_Y^α is normal in $P_Y \geq P_X \geq L_X$ and since $Z = Z(L_X)^\circ$ induces the full group of scalars on Q_X. So either $Q_X \leq K_Y^\alpha$ or $Q_X K_Y^\alpha / K_Y^\alpha \cong Q_X/(Q_X \cap K_Y^\alpha) \cong Q_X/Q'_X = Q_X$ has dimension $2(m-1)$; the latter is impossible (it has dimension 0 or 1). So $[V, Q_Y] = [V, Q_X] \leq [V, K_Y^\alpha]$. But the weight space $V_{\lambda-\alpha}$ appears in $[V, Q_Y]$ and not in $[V, K_Y^\alpha]$. This is a contradiction, so in fact $\langle \lambda, \alpha \rangle = 0$ for α not adjoining $\Pi(L_Y)$.

Now assume $\langle \lambda, \alpha \rangle \neq 0$ for some $\alpha \in \Pi(Y) - \Pi(L_Y)$ such that $\langle \alpha, \Pi(L_i) \rangle = 0$. By the above, there is some $j \neq i$ such that $\langle \alpha, \Pi(L_j) \rangle \neq 0$. The weight $\lambda - \alpha$ is a weight in $[V, Q_Y] = [V, Q_X]$ but not in $[V, K_Y^\alpha]$, so $Q_X \leq K_Y^\alpha$ gives a contradiction as above. So $Q_X \not\leq K_Y^\alpha$. Then $Q_X K_Y^\alpha / K_Y^\alpha \neq 0$ and as above $Q_X K_Y^\alpha / K_Y^\alpha \cong Q_X/(Q_X \cap K_Y^\alpha) \cong Q_X/K^{\beta_1}$ is an irreducible L'_X-module of dimension $2(m-1)$. So $Q_Y/K_Y^\alpha \cong V_{L'_Y}(-\alpha)$ is an irreducible L'_Y-module of dimension $\geq 2(m-1)$.

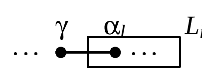
Let $\mu = \lambda - \alpha$. Then $\mu|_{T_i} = \lambda|_{T_i}$ (where T_i is a maximal torus of L_i); and $V_{L'_Y}(\mu) = V_{L_i}(\lambda) \otimes V_{L'_Y}(-\alpha)$, of dimension $\geq \dim(V^1(Q_Y))(2m-2)$. But by Lemma 2.9, $\dim(V^2(Q_Y)) \leq \dim(V^1(Q_Y))(2m-2)$, and $V_{L'_Y}(\mu) \leq V^2(Q_Y)$. This forces $V_\alpha(Q_Y)(= V_{L_Y}(\mu)) = V^2(Q_Y)$; but there is some $\varepsilon \in \Pi(Y) - \Pi(L_i)$ such that $\langle \varepsilon, \Pi(L_i) \rangle \neq 0$, and $V_\varepsilon(Q_Y) \neq 0$. From this contradiction, we have that in fact $\langle \lambda, \alpha \rangle = 0$ for all α not in or adjoining $\Pi(L_i)$. □

We now look at each of the cases 1 – 4 in turn.

4.1. **Case 1.**

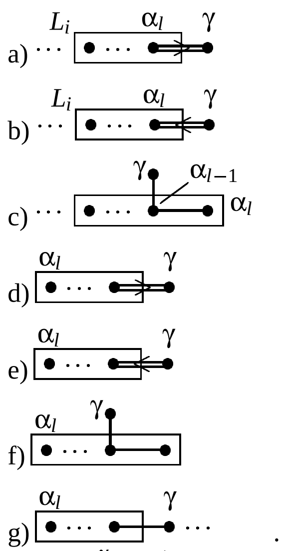
Claim 4.3. *$V^1(Q_Y) \not\cong$ the natural module for L_i, so case 1 on page 20 does not arise.*

Proof. Assume $V^1(Q_Y)$ is isomorphic to the natural module W_i for L_i.

There are several possibilities for the type of L_i. Call the node in L_i with a non-zero label α_l. If L_i is of type B_k, C_k, or D_k, then it corresponds to a subdiagram of the Dynkin diagram for Y at the "end" of that diagram; the picture is: 

. The only node in $\Pi(Y)$ adjoining L_i adjoins α_l; call it γ. Now $W_i \cong V^1(Q_Y) = V_1^1(Q_X) \oplus V_2^1(Q_X)$, and these are the only two L_X-submodules of $V^1(Q_Y)$. They are interchanged by t. Also, Q_Y/K_Y^γ is an L'_Y-module of high weight $-\gamma$; $Q_X K_Y^\gamma / K_Y^\gamma$ is a non-zero t-stable L_X-submodule ($Q_X \leq K_Y^\gamma$ gives a contradiction as in the proof of the last lemma). But we just said that the natural module for L_i has no t-stable L_X-submodules; this forces there to be an L_j adjoining γ such that L'_X projects non-trivially to L_j. Thus the natural module for L_j has dimension $\geq 2(m-1)$. But then if γ has non-zero label, we have a composition factor of high weight $(\lambda - \gamma)|_{T_{L'_Y}}$ in $V^2(Q_Y)$, of dimension $> 2(m-1)\dim(V^1(Q_Y))$. And if γ has label 0, we have $\lambda - \gamma - \alpha_l$, also giving dimension $> 2(m-1)\dim(V^1(Q_Y))$. By Lemma 2.9, this is a contradiction. Note that this same argument holds whenever there is a node in $\Pi(Y) - \Pi(L_i)$ adjoining α_l via a single bond. So L_i must be of type A_k, with no node outside $\Pi(L_i)$ adjoining α_l via a single bond.

By Lemma 4.3, all nodes in $\Pi(Y)$ not adjoining $\Pi(L_i)$ have marking 0. There is no node outside $\Pi(L_i)$ adjoining α_l via a single bond. The other possibilities for nodes adjoining L_i are the following, which we will consider in turn:



a) $Q_Y/K_Y^{\gamma} \cong V^1(Q_Y)$; as above, this forces there to be an L_j adjoining γ, which is absurd.

b) The rank of L_i is k, so $\dim(V^1(Q_Y)) = k+1$. We then have $k+1 = \dim(V^1(Q_Y)) \geq 2^{m-1} > \dim(Q^{\beta_1})$. (The dimension of $V^1(Q_Y)$ is at least 2^{m-1} since the high weight of $V_1^1(Q_X)$ is not symmetric with respect to the graph automorphism of D_{m-1}; thus the Weyl group orbit of the $T_{L'_X}$-high weight of $V_1^1(Q_X)$ contains at least 2^{m-2} weights.) In particular, $k+1 \geq 8$ since $m \geq 4$. Note also that $\dim(V^1(Q_Y))$ is even (so k is odd) because it is the sum of two L'_X-modules of equal dimension. If γ has a non-zero label in the marking for λ on Y, then $V^2(Q_Y)$ has a composition factor with high weight $(\lambda-\gamma)|_{T_{L'_Y}}$, giving $\dim(V^2(Q_Y)) \geq \dim(V_{L_i}((\lambda-\gamma)|_{T_Y})) = \frac{1}{6}(\dim(V^1(Q_Y))+2)(\dim(V^1(Q_Y))+1)(\dim(V^1(Q_Y))) > 2(m-1)\dim(V^1(Q_Y))$, which is a contradiction to Lemma 2.9. If γ has label 0, then the high weight $(\lambda-\alpha_l-\gamma)|_{T_{L'_Y}} = (\lambda_{l-1}+\lambda_l)|_{T_{L'_Y}}$ appears. This weight has $k(k+1)$ conjugates, and the subdominant weight (λ_{l-2}) has $\frac{(k+1)!}{6(k-2)!}$ conjugates. But for $k>6$, $k(k+1)+\frac{(k+1)!}{6(k-2)!} > (k+1)^2 > 2(m-1)\dim(V^1(Q_Y))$, again a contradiction. So b) does not arise.

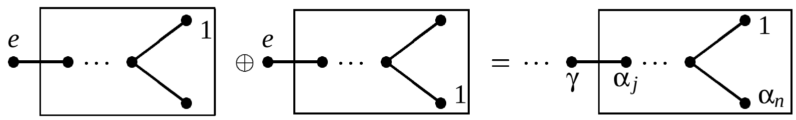
c) Take k as in b). If γ has non-zero marking, then $V^2(Q_Y)$ has a composition factor of high weight $(\lambda-\gamma)|_{T_{L'_Y}} = (\lambda_{l-1}+\lambda_l)|_{T_{L'_Y}}$, and we have a contradiction as above. If γ has marking 0, then we have the high weight $(\lambda-\alpha_l-\alpha_{l-1}-\gamma)|_{T_{L'_Y}} = \lambda_{l-2}|_{T_{L'_Y}}$, which has, as above, $\frac{(k+1)k(k-1)}{6}$ conjugates. For $k \geq 8$, $\frac{(k+1)k(k-1)}{6} > (k+1)^2 \geq 2(m-1)\dim(V^1(Q_Y))$; this is a contradiction to the same result. If $k=7$, then $\dim(V_{L_i}(\lambda_{l-2})) = \binom{8}{3} = 56 > 6\cdot 8 = \dim(Q^{\beta_1})\dim(V^1(Q_Y))$, and again we have a contradiction.

d)-g) Let $k+1 = \dim(V^1(Q_Y))$ as above. In any of these cases, if γ has a non-zero label, then $V^2(Q_Y)$ has a composition factor (given by the high weight $(\lambda-\gamma)|_{T_{L'_Y}}$) of dimension greater than $\dim(Q^{\beta_1})\dim(V^1(Q_Y))$, giving a contradiction as above. If γ has a zero label, then $V \cong$ the natural module W for Y, which is impossible, since X acts irreducibly on W but not on V. $\square$

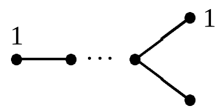
4.2. **Case 2.**

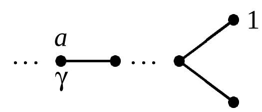
Claim 4.4. *Case 2 on page 20 does not arise.*

Proof. We have the picture below, with the boxed diagrams on the left of type D_{m-1} and on the right of type D_m.:



with $j = n - m + 1$ (n is the rank of Y, as always). If γ has a non-zero label, then in the characteristic 0 case, $\dim(V^2(Q_Y)) \geq \dim(V_\gamma(Q_Y)) \geq \dim(V_{D_m}(\lambda_j + \lambda_{n-1})) = (2m - 1)\dim(V^1(Q_Y)) > \dim(Q^{\beta_1})\dim(V^1(Q_Y))$, again a contradiction to Lemma 2.9. The only problem here is when the irreducible module $V_{D_m}(\lambda_j + \lambda_{n-1})$ is not the Weyl module.

We can use the Andersen-Jantzen sum formula to check that in those characteristics for which the Weyl module does reduce, it reduces only by 2^{m-1}, making the bound sharp: In these cases, $\dim(V_{D_m}(\lambda_j + \lambda_{n-1})) = (2m-2)\dim(V^1(Q_Y))$. So if there is anything else in $V_\gamma(Q_Y)$, $V^2(Q_Y)$ will again be too big. So assume the D_m-module V^1 with high weight marking 

(of high weight $\lambda_j + \lambda_{n-1} = \lambda'$) does reduce. Then the dimension of the weight space $V^1_{\lambda' - \alpha_j - \cdots - \alpha_{n-1}}$ in $V(\lambda')$ is $m-2$ (this dimension is $m-1$ in the characteristic 0 case). But in V, which has marking 

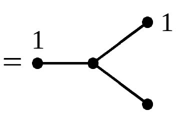
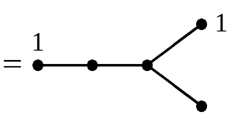
, the dimension of the weight space for the weight $\lambda - \gamma - \alpha_j - \cdots - \alpha_{n-1}$ is m or $m-1$; and this weight space is in $V_\gamma(Q_Y)$. So if $V(\lambda')$ reduces, then there is something else in $V_\gamma(Q_Y)$, and again we get that $V^2(Q_Y)$ is too large. So in fact γ has label 0 (i.e. $\langle \lambda, \gamma \rangle = 0$), and, since we have already shown that nodes not adjoining L_i have 0 label, V is the spin module for Y.

Now look at a different parabolic subgroup of X: Let $P_X = L_X Q_X$ correspond to $\Pi(X) - \{\beta_{m-1}, \beta_m\}$. Use the standard notation for a basis of the Lie algebra of X: For a simple Lie algebra with root system Φ having basis $\{\alpha_1, \ldots, \alpha_m\}$, use the Chevalley basis $\{e_\alpha, f_\alpha, h_i | \alpha \in \Phi^+, 1 \leq i \leq m\}$, satisfying the usual relations — in particular, $[e_{\alpha_i} f_{\alpha_i}] = h_i$. Then each V_i ($i = 1,2$) is spanned by vectors of the form $w = f_\alpha \ldots f_\beta v_i^+$, where v_i^+ is a maximal vector. Order the roots in $\Sigma(X)$ so that the last f_γ applied correspond to roots in $\Sigma(X) - \Sigma(L'_X)$. Now $V_i^2(Q_X)$ is spanned by vectors $f_\alpha w$ where $w = f_\beta \ldots f_\varepsilon v_i^+$ such that all the roots $\beta, \ldots, \varepsilon$ are in $\Sigma(L'_X)$ and α has Q_X-level 1 (i.e. α has β_{m-1}- or β_m-coefficient -1 and the other 0). If we take a maximal linearly independent set of such w, we have a basis of $V_i^1(Q_X)$; and there are $2(m-1)$ roots α of Q_X-level 1. So $\dim(V_i^2(Q_X)) \leq 2(m-1)\dim(V_i^1(Q_X))$. This gives $\dim(V^2(Q_Y)) \leq 2\dim(V_1^2(Q_X)) \leq 2(2(m-1)\dim(V_1^1(Q_X))) = 2(m-1)\dim(V^1(Q_Y))$.

If $e = 0$ (with e the labelling on β_1 as in the picture at the beginning of the proof), then $V|_X$ is a sum of two spin modules. So $\dim(V) = 2(2^{m-1}) = 2^m$. Since V is a spin module for Y, $\dim(V) = 2^{n-1}$. So $X = D_{n-1}$. But the natural module for Y has dimension $2n$, and D_{n-1} for $n \geq 5$ has no irreducible restricted modules of this dimension. So $e \neq 0$.

V is the spin module for Y, and the parabolic subgroup P_Y of Y in which we embed this new P_X must contain both of the root groups corresponding to the node with a label of 1 in the marking for the Y-high weight of V (since $e \neq 0$). Since this 1 is the only non-zero label in the marking, the L_i which contains it is the only L_j acting non-trivially on $V/[V,Q_Y]$. Now L'_X is of type A_{m-2}, and $V_1/[V_1,Q_X]$ has high weight $e\delta_1$.

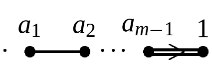
There are two possibilities for the type of L_i: D_{l+1} for some $l \geq 3$, and A_l. If L_i is of type D_{l+1}, then $\dim(V/[V,Q_Y]) = 2^l$ is a power of 2. By Lemma 2.5, $\dim(V_1/[V_1,Q_X]) = \binom{m-2+e}{e}$, which is not a power of two unless $e = 1$ or $m = 3$. But $m \geq 4$, so this forces $e = 1$. Then $\dim(V/[V,Q_Y]) = (m-2)+1 = 2^l$. A group of type D_{l+1} for $l \geq 3$ does not contain a group of type A_{2^l-1}, however, so L_i is not of type D_{l+1} for $l \geq 3$.

If L_i is of type A_l, then by Lemma 2.5 we have $l+1 = 2\dim(V_1/[V_1,Q_X]) = 2\binom{m-2+e}{e} > 4$. The root α_n (see the picture at the beginning of the proof) is not contained in the set of roots corresponding to L_Y (since $l > 3$), so $V^2(Q_Y)$ contains a composition factor of high weight $(\lambda - \alpha_{n-1} - \alpha_{n-2} - \alpha_n)|_{T_Y}$, of dimension $\binom{l+1}{3}$. Then we have $\frac{1}{6}(l+1)l(l-1) \leq \dim(V^2(Q_Y)) \leq 2(m-1)\dim(V^1(Q_Y)) = 2(m-1)(l+1)$, with l given in terms of m and e above. This is impossible in all cases except $m = 4, e = 1$ and $m = 5, e = 1$. So the only two possibilities left here are $V_1|_X =$  and $V_1|_X =$  .

Remember that V is the spin module for Y, so has dimension a power of 2. So V_1 has dimension a power of 2. The only time when one of the above two modules has dimension a power of 2 is the D_5 module of high weight $\delta_1 + \delta_4$ in characteristic 5, of dimension 2^7. So $\dim(V) = 2^8$ and Y is of type D_9. But D_5 has no 18-dimensional irreducible representations in characteristic 5. So L_i is not of type A_l. □

4.3. **Case 3.**

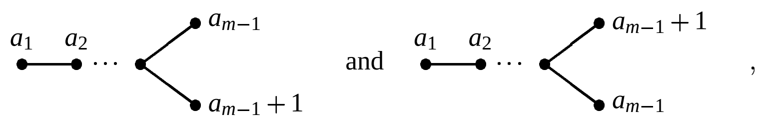
Claim 4.5. *The situation outlined in case 3 on page 20 does not arise.*

Proof. Here V has high weight marking $\cdots$ , restricting to D_{m-1} and with $a_2, \dots, a_m = 1$ related as in [4, 3.3], and with labels to the left of a_1 all 0. Let P_X be the parabolic subgroup of X corresponding to $\Pi(X) - \{\beta_m\}$. As always, embed P_X in a parabolic subgroup P_Y of Y via the construction given in the introduction, so that $Q_X \leq Q_Y$ and $L_X \leq L_Y$.

As in the proof of Lemma 3.5, we show that in this case $V/[V,Q_Y]$ is irreducible as an L_X-module: Let $Z = Z(L_X)^\circ$. By construction, $Z \leq Z(L_Y)$ (where $L_Y = C_Y(Z)$), so Z induces scalars on $V/[V,Q_Y]$ (since L_Y acts irreducibly). But if $V/[V,Q_Y]$ is not irreducible for L_X, then $V/[V,Q_Y] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X]$, and Z acts differently on these two L_X-modules:

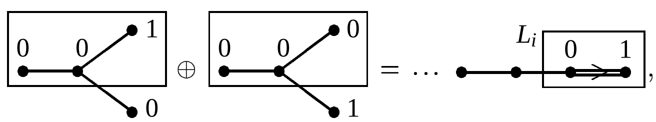
$$\begin{aligned} Z &= \{\mathrm{diag}(a,\dots,a,a^{-1},\dots,a^{-1}) \mid a \in K^*\} \\ &= \{\mathrm{diag}(a^2,\dots,a^2,a^{-2},\dots,a^{-2}) \mid a \in K^*\} \\ &= \{h_{\beta_1}(a^2)h_{\beta_2}(a^4)\dots h_{\beta_{m-2}}(a^{2(m-2)})h_{\beta_{m-1}}(a^{m-2})h_{\beta_m}(a^m) \mid a \in K^*\}. \end{aligned}$$

The two X-modules V_1, V_2 have high weight labelling



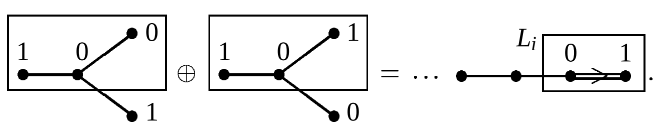
so $h_{\beta_1}(a^2)h_{\beta_2}(a^4)\ldots h_{\beta_{m-2}}(a^{2(m-2)})h_{\beta_{m-1}}(a^{m-2})h_{\beta_m}(a^m)$ acts differently on a high weight vector $v_1 \in V_1$ than on a high weight vector $v_2 \in V_2$. Since v_i has a non-zero image in $V_i/[V_i, Q_X]$, this shows that only one of the $V_i/[V_i, Q_X]$ can be in $V/[V, Q_Y]$. So $V/[V, Q_Y]$ is irreducible as an L_X-module. Assume V_1 is the summand which projects non-trivially to $V/[V, Q_Y]$ (so $V_2 \subseteq [V, Q_Y]$).

Irreducible restricted A_{m-2}-modules are tensor indecomposable (Lemma 2.1), so only one of the simple factors L_i of L_Y' acts non-trivially on $V/[V, Q_Y] = V^1(Q_Y)$. The two possibilities for the type of L_i are B_k and A_k. If L_i is of type B_k, then $V^1(Q_Y) \cong$ the natural module for L_i, since there are no overgroups of A_l of type B_k appearing in [9, Table 1]. But we know $a_m = 1$, so the only possibility here is L_i of type $B_2 \cong C_2$, and $V^1(Q_Y) \cong$ the natural module for C_2, of dimension 4. Then $m = 4$, $L_X' = A_3$, $X = D_4$, and $V_1^1(Q_X)$ is of L_X-high weight δ_3 or δ_1. If δ_3, then the picture is



so $\dim(V) = \dim(V_1) + \dim(V_2) = 16$. Since $a_m = 1$, the T_Y-high weight of V has at least $|\mathcal{W}|/|\mathcal{W}_{A_{n-1}}| = 2^n n!/n! = 2^n$ conjugates, where $\mathcal{W}$ is the Weyl group of type B_n, and $\mathcal{W}_{A_{n-1}}$ that of type A_{n-1}. So $n \leq 4$; since $4 = m \leq n$, we have $Y = B_4$. But then if $p \neq 2$, $\dim(W) = 9$, and D_4 has no irreducible representations of dimension 9. If $p = 2$, then we have one of the situations studied already in [4], which does give the example listed in the theorem.

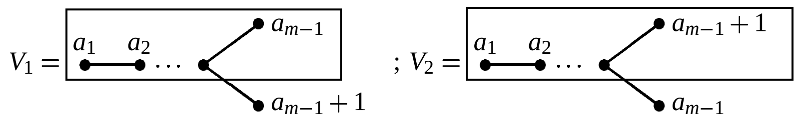
If the $T_{L_X'}$-high weight of $V_1/[V_1, Q_X]$ is δ_1, then the picture is



In this case, since $\dim(V) = 2\dim(V_1)$, we have $\dim(V) = 112$ if $p \neq 2$, $\dim(V) = 96$ if $p = 2$. But Y cannot have type B_4 unless $p = 2$ (in which case [4] tells us there are no examples of this type) as above, and no irreducible restricted B_l-module, for $l \geq 5$, whose high weight has an λ_l-coefficient of 1, has dimension 112 or 96. So L_i is not of type B_k.

So L_i is of type A_k. The arguments in Lemma 4.3 showing that nodes not adjoining L_i have marking 0 fail here, since $[V, Q_Y] \neq [V, Q_X]$. So we need new arguments. Assume γ is a node in $\Pi(Y)$ which has non-zero label and does not adjoin $\Pi(L_Y)$. The argument in the proof of Lemma 4.2 that $Q_X \leq K_Y^\gamma$ is still valid here, so $V_{\lambda-\gamma} \notin [V, Q_X]$. We have $V/[V, Q_Y] = (V_1 \oplus V_2)/(V_2 \oplus [V, Q_X])$, so if we can show that $V_{\lambda-\gamma} \notin V_2$ we will have a contradiction, since $V_{\lambda-\gamma} \in [V, Q_Y]$.

Since $\langle \gamma, \Pi(L_Y) \rangle = 0$, we know that $\lambda|_{T_{L'_Y}} = (\lambda - \gamma)|_{T_{L'_Y}}$, thus $\lambda|_{T_{L'_X}} = (\lambda - \gamma)|_{T_{L'_X}}$. If γ_i is the T_X-high weight of V_i, then $\lambda|_{T_X} = \gamma_i$ for $i = 1$ or 2; we have the pictures:



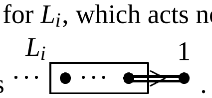
Assume $\lambda|_{T_X} = \gamma_1$. Then $\gamma_1|_{T_{L'_X}} = \lambda|_{T_{L'_X}} = (\lambda - \gamma)|_{T_{L'_X}}$ is the $T_{L'_X}$-high weight of $V_1/[V_1, Q_X]$. Then to have the weight $(\lambda - \gamma)|_{T_X}$ in V_2 would imply we could subtract roots from γ_2 and obtain a weight ε which has the same labelling as γ_1 on $\Pi(L_X)$. Say we subtract $c\beta_1$. To balance this (the labelling on β_1 is the same for γ_1 as for γ_2), we must subtract $2c\beta_2, 3c\beta_3, \ldots, (m-2)c\beta_{m-2}$. Say we subtract $d\beta_{m-1}$ and $e\beta_m$. To balance the δ_{m-2} coefficient, we need $2(m-2)c - (m-3)c - d - e = 0$. To get a_{m-1} as the δ_{m-1} coefficient, we need $2d - (m-2)c = 1$. These give $e = (m-1)c - d$ and $d = ((m-2)c+1)/2$.

Now [9, 3.6 iv)] gives $\gamma|_Z = \beta_m|_Z$. So since $Z \leq T_X$ and $(\lambda - \gamma)|_{T_X} = (\gamma_1 - \gamma)|_{T_X}$, we have $\varepsilon|_Z = (\lambda - \gamma)|_Z = (\gamma_1 - \gamma)|_Z = (\gamma_1 - \beta_m)|_Z$. We computed Z above and

$$(\gamma_1 - \beta_m)(h_{\beta_1}(a^2)h_{\beta_2}(a^4)\ldots h_{\beta_{m-2}}(a^{2(m-2)})h_{\beta_{m-1}}(a^{m-2})h_{\beta_m}(a^m)) = a^{A-4}$$

where $A = 2a_1 + 4a_2 + \cdots + 2a_{m-2}(m-2) + a_{m-1}(m-2) + (a_{m-1}+1)m$. So ε must act on Z as a^{A-4}. Let the δ_m-coefficient of ε be a'_m; we know all the other coefficients of ε (they are $a_1, \ldots, a_{m-1}$). Since ε must act as a^{A-4} on the element of Z given above, we have $A - 4 = 2a_1 + 4a_2 + \cdots + 2a_{m-2}(m-2) + a_{m-1}(m-2) + a'_m m$, or $a'_m m = (a_{m-1}+1)m - 4$. But $a'_m = a_{m-1} - 2e + (m-2)c$ (we subtracted $(m-1)c\beta_{m-2}$ and $e\beta_m$). So $1 - (4/m) = -2e + (m-2)c = 2c(m-2) - 2c(m-1) + 1$. This gives $4/m = 2c$, or $2/m = c$. But $m \geq 4$, so $2/m = c$ has no integer solutions. This is a contradiction.

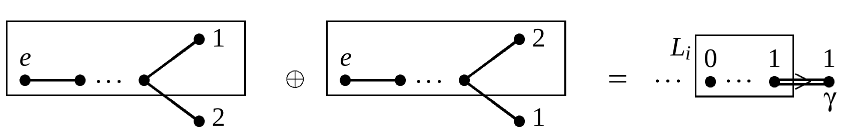
If $\lambda|_{T_X} = \gamma_2$, similar computations also give $2/m = c$ and a contradiction as above. So all nodes with non-zero labels must be in $\Pi(L_i)$ or adjoin $\Pi(L_Y)$.

Now let $\gamma = \alpha_n$ be the "end" node of the Dynkin diagram for Y, which has type B_n. We know α_n has a label of 1 (and $\alpha_n \notin \Pi(L_i)$ because L_i does not have type B_l); so, by the above, α_n adjoins some $\Pi(L_j)$. If $j \neq i$, then we may assume L_X projects non-trivially to L_j (else we can take P_Y such that $L'_Y = L_1 \times \cdots \times \widehat{L_j} \times \ldots$ and still have a parabolic satisfying all the hypotheses we used above; then α_n would not adjoin $\Pi(L_Y)$ and the above would give a contradiction). So L_j has rank at least $m-1$; but then all labels to the left of L_j are 0 and there is no room for L_i, which acts non-trivially on $V/[V, Q_Y]$. So in fact α_n adjoins $\Pi(L_i)$. So the picture is 

.

The possibilities are: $L'_X, L'_Y, V/[V, Q_Y]$ appear in Table 1 of [9]; $V/[V, Q_Y]$ is the natural module for L_i; or L_i is of type A_{m-1}, with the embedding $L'_X \hookrightarrow L_i$ an isomorphism. There are only 2 possibilities arising from Table 1 of [9]; they give easy contradictions to the fact that most of the (possibly) non-zero labels for the T_X-high weights of V coincide with the corresponding labels for the T_Y-high weight of V.

Assume that $V^1(Q_Y)$ is isomorphc to the natural module for L_i. Let k be the rank of L_i. If the picture is



then we know that $p = 5$ (as this is the only characteristic in which this configuration arises in [4, 3.3]) and that there are no non-zero labels to the left of L_i (since at most the end m labels are non-zero as mentioned at the beginning of the proof, and $k > m$). Let V_1 be the L_X-module which projects non-trivially to $V/[V,Q_Y]$ (not necessarily the first summand pictured above). Here $Q_X \le K_Y^{\alpha_n}$ (since otherwise $Q_Y/K_Y^{\alpha_n} \cong V/[V,Q_Y]$ has an L_X-submodule $Q_X K_Y^{\alpha_n}/K_Y^{\alpha_n} \cong Q_X/K^{\beta_m}$, which it does not since $V/[V,Q_Y]$ is irreducible for L'_X), so again we will have a contradiction if we can show that $V_{\lambda-\alpha_n} \not\subseteq V_2$, since $[V,Q_Y] = [V_1,Q_X]+V_2$, $V_{\lambda-\alpha_n} \not\subseteq [V_1,Q_X]$, and $V_{\lambda-\alpha_n} \subseteq [V,Q_Y]$.

Call the two T_X-high weights pictured above μ and ν respectively. If $e = 0$, we know from the main result of [4] that $\dim(V_1)+\dim(V_2)$ is the dimension of the B_m-module (where $m =$ the rank of $X = D_m$) with high weight $\lambda_{m-1}+\lambda_m$. So if $n(=$ the rank of $Y)$ is greater than m, the dimension of V is strictly larger than $\dim(V_1)+\dim(V_2)$, which is a contradiction. So if $e = 0$, then $n = m$ and we must be in the $D_n < B_n$ case. Assume $e > 0$. We have $(\lambda-\alpha_n)|_{T_{L'_Y}} = \lambda|_{T_{L'_Y}} + \lambda_{n-1}|_{T_{L'_Y}} = 2\lambda|_{T_{L'_Y}}$. Assume $\lambda|_{T_X} = \mu$ (i.e. the first summand pictured above is the one that projects non-trivially to $V/[V,Q_Y]$). Then $\mu|_{T_{L'_X}} = \lambda|_{T_{L'_X}}$ is the $T_{L'_X}$-high weight of $V/[V,Q_Y]$. To have the weight $(\lambda-\alpha_n)|_{T_X}$ in V_2 would mean we could subtract roots from ν and obtain a weight ε which restricts to $T_{L'_X}$ in the same way as 2λ does; i.e. there must be integers $c,\dots,d,f$ such that $2e\delta_1+2\delta_{m-1}+a\delta_m = e\delta_1+2\delta_{m-1}+\delta_m-(c\beta_1+\cdots+d\beta_{m-1}+f\beta_m)$ for some a. If we subtract $c\beta_1$, we must subtract $(2c+e)\beta_2,(3c+2e)\beta_3,\dots,((m-2)c+(m-3)e)\beta_{m-2}$. Say we subtract $d\beta_{m-1}$ and $f\beta_m$. Then since the coefficients of $\delta_{m-2},\delta_{m-1}$ remain unchanged, we have $(m-3)c+(m-4)e+d+f = 2((m-2)c+(m-3)e))$ and $(m-2)c+(m-3)e = 2d$. These together give $f = (m(c+e)-e)/2$

Lemma 3.6 iv) in [9] gives $\alpha_n|_Z = \beta_m|_Z$. So $(\lambda-\alpha_n)|_Z = (\mu-\beta_m)|_Z$. We computed Z above, and

$$(\mu-\beta_m)(h_{\beta_1}(a^2)h_{\beta_2}(a^4)\dots h_{\beta_{m-2}}(a^{2(m-2)})h_{\beta_{m-1}}(a^{m-2})h_{\beta_m}(a^m)) \\ = a^{2e+(m-2)+2m-4}.$$

So ε must act on Z as $a^{2e+(m-2)+2m-4}$. But 2λ acts as

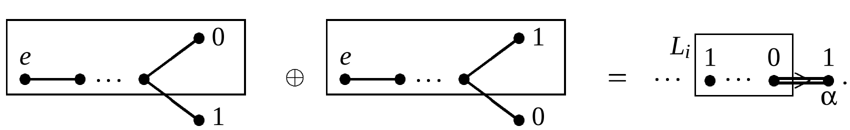
$$a^{4e+2(m-2)+(1-2f+(m-2)c+(m-3)e)m}.$$

So these two exponents must be equal. Using the above expression for f, this simplifies to $m(e+c) = e+2$, which has no solutions in non-negative integers with $m \ge 4$. So $\lambda|_{T_X} \ne \mu$.

If $\lambda|_{T_X} = \nu$, then similar calculations give $2f = m(c+e)-e-3$ and $m(e+c-3) = e$; this system also has no solutions in non-negative integers with $m \ge 4, e \ge 1$.

So the picture above does not occur.

If the non-zero label is on the other end of L_i, then the picture is

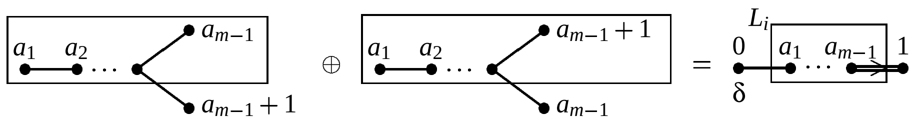


The rank of L_i is $\dim(V^1(Q_Y))-1$; we know that the node in the Dynkin diagram for Y with a label of 1 must be within m nodes of the end (it must be α_i for $i \ge n-m$), as noted at the beginning of this proof. If the second summand in the picture is the one that projects non-trivially to $V^1(Q_Y)$, this forces $e = 0$; but then V has dimension too large to be the sum of the two spin modules for D_m. If the first summand projects non-trivially, $e = 1$ for the same reason and L_i has rank $m-1$.

If there is a node to the left of L_i (i.e. if $n > m$), let P_Y be the parabolic subgroup of Y corresponding to the end m nodes of the Dynkin diagram. Then $V|_{L'_Y}$ has a composition factor (not all of V) which is isomorphic to the B_m module with high weight $\lambda_1 + \lambda_m$. This B_m-module has D_m-high weights $\delta_1 + \delta_{m-1}$ and $\delta_1 + \delta_m$ (for the $D_m \leq L'_Y$ in the natural way); thus V is too large to be the sum of two D_m-modules of these high weights. So this case does not occur. If there is no node to the left of L_i then $Y = B_m$ and we are in one of the cases studied in [4], which tells us there are no examples.

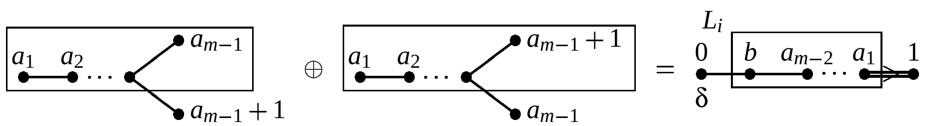
So we are left with the case when the embedding $L'_X \hookrightarrow L_i$ is an isomorphism, so L_i is of type A_{m-1} and the labelling of the high weight of $V/[V,Q_Y]$ on $\Pi(L_i)$ is the same as the labelling on $\Pi(L'_X)$. The two possible pictures are:

(1)



with the first summand projecting non-trivially to $V^1(Q_Y)$; or

(2)



with $b = a_{m-1}$ if the first summand projects non-trivially to $V/[V,Q_Y]$, and $b = a_{m-1}+1$ if it is the second summand which projects non-trivially (as noted at the beginning of this proof, all labels to the left of the highlighted nodes of the Dynkin diagram for Y in the pictures above are 0).

If we have picture (1) above, then the question is whether or not δ exists. The B_m-module with high weight $a_1\lambda_1 + \cdots + a_{m-1}\lambda_{m-1} + \lambda_m$ (where the λ_i are for the moment the fundamental dominant weights of this Levi factor of Y of type B_m), when considered as a module for the D_m sitting in B_m in the usual way, has high weights $a_1\delta_1 + \cdots + a_{m-1}\delta_{m-1} + (a_{m-1}+1)\delta_m$ and $a_1\delta_1 + \cdots + (a_{m-1}+1)\delta_{m-1} + a_{m-1}\delta_n$ (this was noted in the proof of [4, 3.3]). So if δ exists, then V is too large to be the sum of two D_m-modules of these high weights. So Y in fact has type B_m; but then we are back in the situation of [4], which tells us the only possibility here is the one in the statement of the theorem.

If we have picture (2) above, then the relationship between the a_i which we know from [4] tells us that $a_2 = a_{m-2}, a_3 = a_{m-3}, \ldots$, and therefore $a_1 = a_{m-1}$. If the first summand projects non-trivially to $V^1(Q_Y)$, then we in fact have an instance of picture (2), which we covered above.

So we may assume we have picture (3), with the second summand projecting non-trivially, i.e. the isomorphism $L'_X \to L_i$ is given by the graph isomorphism sending $\beta_1 \to \alpha_{m-1}$, $\beta_2 \to \alpha_{m-2}$, ... , $\beta_{m-1} \to \alpha_1$ (here and below, the α_i are the fundamental roots of Y corresponding to the "end" m nodes of the Dynkin diagram, with the λ_i the corresponding fundamental dominant weights).

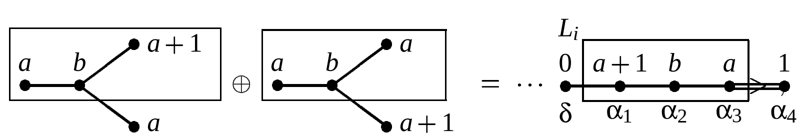
We now define a normal subgroup of P_Y which is the "one level down" analogue of K_Y^γ (see Lemma 2.10 and the paragraph preceeding it). For a root $\gamma \in \Pi(Y) - \Pi(L_Y)$, let $\Sigma_Y(\gamma)$ be the set of roots in $\Sigma^-(Y)$ which have γ-coefficient -1 and coefficient 0 for other fundamental roots not in $\Pi(L_Y)$. Then as in the introduction, K_Y^{γ} is the product of root groups U_β for $\beta \in (\Sigma^-(Y) - \Sigma^-(L_Y) - \Sigma_Y(\gamma))$. Analogously, let $\Sigma_Y(2\gamma)$ be the set of roots in $\Sigma^-(Y)$ having γ-coefficient -2 and coefficient 0 for other roots in $\Pi(Y) - \Pi(L_Y)$. Then let $K_Y^{2\gamma}$ be the product of root groups U_β for $\beta \in (\Sigma^-(Y) - \Sigma^-(L_Y) - \Sigma_Y(\gamma) - \Sigma_Y(2\gamma))$. $K_Y^{2\gamma}$ is normal in P_Y by the commutator relations, and $K_Y^{2\gamma} \leq K_Y^{\gamma}$. We have previously considered the quotient Q_Y/K_Y^{γ}; now we wish to look at $K_Y^{\gamma}/K_Y^{2\gamma}$. Let $\gamma = \alpha_m$ be the short fundamental root of Y. Then $K_Y^{\alpha_m}/K_Y^{2\alpha_m} \cong$ the irreducible L_i-module with high weight $-(2\alpha_m + \alpha_{m-1})|_{T_{L_i}} = \lambda_{m-2}|_{T_{L_i}}$.

Consider $Q^{\beta_m} = Q_X/K^{\beta_m}$, where K^{β_m} is the product of those T_X-root groups U_γ for $\gamma \in \Sigma^-(X)$ with β_m-coefficient less than -1 (which in this case is trivial, as there are no such roots in a root system of type D_m). By Lemma 2.10, Q^{β_m} is an L'_X-module with high weight $(-\beta_m)|_{T'_{L_X}} = \delta_{m-2}|_{T'_{L_X}}$ and thus dimension $\binom{m}{2}$.

By [9, 3.1], $Q_Y/K_Y^{\alpha_m}$ is an irreducible L'_Y-module with high weight $-\alpha_m|_{T_{L'_Y}} = \lambda_{m-1}|_{T_{L'_Y}}$, which implies that $Q_Y/K_Y^{\alpha_m}$ has dimension m. It has an L_X-submodule $Q_X K_Y^{\alpha_m}/K_Y^{\alpha_m}$ (by [9, 3.3]). If $Q_X \not\leq K_Y^{\alpha_m}$, then $Q_X K_Y^{\alpha_m}/K_Y^{\alpha_m} \cong Q_X/(Q_X \cap K_Y^{\alpha_m}) \cong Q_X/Q'_X \cong Q_X/K^{\beta_m}$ has dimension $\binom{m}{2} > m$, which is impossible. So $Q_X \leq K_Y^{\alpha_m}$. Then $K^{\beta_m} = Q'_X \leq K_Y^{\alpha_m\prime} \leq K_Y^{2\alpha_m}$, and we can project $Q_X/K^{\beta_m} \longrightarrow K_Y^{\alpha_m}/K_Y^{2\alpha_m}$. If this map has non-zero image then it is in fact an isomorphism, since Q_X/K^{β_m} is an irreducible L'_X-module and $\dim(Q_X/K^{\beta_m}) = \dim(K_Y^{\alpha_m}/K_Y^{2\alpha_m})$. But then $K_Y^{\alpha_m}/K_Y^{2\alpha_m}$ has L'_X-high weight δ_{m-2}, whereas we know, by the fact that the isomorphism $L'_X \to L_i$ is given by the graph isomorphism sending $\beta_1 \to \alpha_{m-1}$, $\beta_2 \to \alpha_{m-2}, \ldots, \beta_{m-1} \to \alpha_1$, that $K_Y^{\alpha_m}/K_Y^{2\alpha_m}$ must have L'_X-high weight δ_2 (since it has L'_Y-high weight λ_{m-2}). So $2 = m-2$, i.e. X is of type D_4.

On the other hand, if $Q_X \leq K_Y^{2\alpha_m}$ (i.e. if the map above is not an isomorphism), then we note that the roots β whose corresponding root subgroups appear in $K_Y^{2\alpha_m}$ all involve some fundamental root in $\Pi(Y) - \Pi(L_Y)$ other than α_m (since $3\alpha_m$ appears in no root). So if we let $P_Y^1 = \langle P_Y, U_{\pm\alpha_m}\rangle$, we have a new parabolic subgroup of Y satisfying $Q_Y^1 \geq Q_X, L_Y^1 \geq L_X, P_Y^1 \geq P_X$. If now $V/[V, Q_Y^1]$ is irreducible as an L'_X-module, then the first part of the proof of this lemma gives a contradiction. If $V/[V, Q_Y^1]$ is the sum of two irreducibles for $L'_X \cong L_i$, then we have a contradiction because this case would have to appear in [4], and it does not.

So we are left with the picture



with the first summand projecting non-trivially to $V^1(Q_Y)$. In fact there are three possibilities: $a = (p-3)/2, b = (p+1)/2$; $a = 0, b = (p-5)/2$; or $a = b = 0$. If $a = b = 0$, then the T_Y-high weight of V has at least 32 conjugates, making V too large to be the sum of two D_4-modules of dimension 8. We may assume $\mathrm{rank}(Y) = n \geq 5$, since otherwise [4] tells us there are no examples.

Suppose $a = 0, b = (p-5)/2$. Then V_1 has D_4-high weight $\frac{p-5}{2}\delta_2 + \delta_4$, and V has dimension at least the dimension of the B_5-module with high weight $\lambda_1 + \frac{p-5}{2}\lambda_2 + \lambda_4$ (the λ_i refer to the fundamental weights corresponding to the end four nodes of the Dynkin diagram for Y). Thus V, as a module for the obvious $B_4 < Y$, has a composition factor V'

of high weight $\lambda_1 + \frac{p-5}{2}\lambda_2 + \lambda_4$, and another, V'', of high weight $\frac{p-3}{2}\lambda_2 + \lambda_4$ (if P_Y is the parabolic subgroup of Y corresponding to the end four nodes, then $V^2(Q_Y)$ has an L_Y-high weight $\lambda - \delta - \alpha_1$, with the above restriction to $T_{L'_Y}$). As a module for the $D_4 < B_4$, V' has high weights $\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4$ and $\delta_1 + \frac{p-5}{2}\delta_2 + \delta_3$, while V'' has high weights $\frac{p-3}{2}\delta_2 + \delta_4$ and $\frac{p-3}{2}\delta_2 + \delta_3$. So $\dim(V) \geq 2\dim(V_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) + 2\dim(V_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4))$. Using the Andersen-Jantzen sum formula ([1, 6]), we see that

$$\dim(V_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) = \dim(W_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) - \dim(W_{D_4}(\delta_1 + \frac{p-9}{2}\delta_2 + \delta_4)),$$

and

$$\dim(V_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4)) = \dim(W_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4)) - \dim(W_{D_4}(\frac{p-9}{2}\delta_2 + \delta_4)),$$

where $W_{D_4}(\beta)$ denotes the Weyl module for D_4 with high weight β (the second terms on the right hand sides of the above equalities appear only if $p - 9 \geq 0$). Calculating these dimensions with the Weyl character formula, we find that

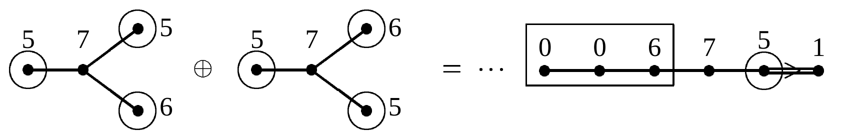
$$\begin{aligned}
\dim(V) &\geq 2(\dim(V_{D_4}(\delta_1 + \frac{p-5}{2}\delta_2 + \delta_4)) + \dim(V_{D_4}(\frac{p-3}{2}\delta_2 + \delta_4))) \\
&> 2\dim(W_{D_4}(\frac{p-5}{2}\delta_2 + \delta_4)) \\
&\geq 2\dim(V_{D_4}(\frac{p-5}{2}\delta_2 + \delta_4)) \\
&= 2\dim(V_1) = \dim(V_1) + \dim(V_2).
\end{aligned}$$

This is a contradiction, so in fact this case does not occur.

Finally, if $a \neq 0 \neq b$, then we take yet another parabolic: Let P_X be the parabolic subgroup of X corresponding to $\{\beta_1, \beta_3, \beta_4\} \subseteq \Pi(X)$. Then L'_X is a product of three A_1's. Here P_X is t-stable, so when we embed P_X in a parabolic subgroup P_Y of Y, as usual, we have $[V, Q_Y] = [V, Q_X]$. We denote $T_{L'_X}$-weights by (a_1, a_3, a_4), where $a_i \in \mathbf{Z}$ is the value of the weight on the torus corresponding to β_i. The high weight of $V_1/[V_1, Q_X]$ is $(a, a+1, a)$; that of $V_2/[V_2, Q_X]$, $(a, a, a+1)$. The weights which appear in $V_1/[V_1, Q_X]$ have the form $(a - 2b_1, a + 1 - 2b_3, a - 2b_4)$, and those in $V_2/[V_2, Q_X]$ have the form $(a - 2c_1, a - 2c_3, a + 1 - 2c_4)$. The weight spaces in $V_i/[V_i, Q_X]$ all have dimension 1, and no weight can appear in both. So the weight spaces of $V/[V, Q_Y]$ have dimension 1.

Those modules for simple algebraic groups which have all weight spaces of dimension 1 are classified in [9, chapter 6]. We have $\dim(V_1/[V_1, Q_X]) = (a+1)^2(a+2)$, so $\dim(V/[V, Q_Y]) = 2(a+1)^2(a+2)$. With the labelling on Y known, we can compare the various possibilities for the factors of L'_Y which act non-trivially on $V^1(Q_Y)$ (remembering that they must appear in [9, 6.1]) with the known dimension of $V^1(Q_Y)$, and we get a contradiction in every case but one: $p = 13$, with the high weights and the embedding of P_X in

P_Y as in this picture:



But now the L'_Y-module Q_Y/K_Y^γ has dimension 2, while Q_X/Q'_X has dimension 8; this forces $Q_X \le K_Y^\gamma$ (otherwise $Q_X K_Y^\gamma/K_Y^\gamma \cong Q_X/Q'_X$ is a submodule of dimension 8 of Q_Y/K_Y^γ, which is impossible); but then $V_{\lambda-\gamma} \notin [V,Q_X] = [V,Q_Y]$, which is absurd since γ has a non-zero label.

So we get no examples here. □

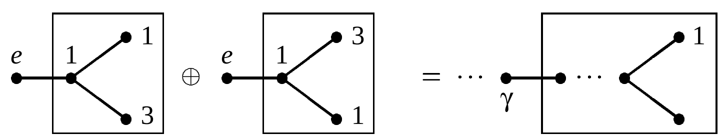
4.4. **Case 4.** We are left with case 4 on page 20.

Lemma 4.6. *If W_i is irreducible for L'_X then $p=2$, $X=D_n$, and $Y=B_n$ or C_n, with V a spin module for Y and a sum of two spin modules for X.*

Proof. We are back to the situation where P_X is the maximal parabolic subgroup corresponding to $\Pi(X)-\{\beta_1\}$. Inductively we need only check the case where $p=2$, $L'_X = D_{m-1}$, and $L_i = B_{m-1}$ or C_{m-1}, with $V/[V,Q_Y]$ a spin module for L_i and a sum of two spin modules for L'_X; and the single case which occurred in section 3: $L_i = D_{10}, (V/[V,Q_Y])|_{L_i} = \mathrm{spin}(L_i), L'_X = D_3 = A_3, V_1/[V_1,Q_X]$ of L_X-high weight $3\delta_3+\delta_2+\delta_1$, $p \neq 2,3,5,7$.

If we are in the first setup, then the arguments of the last subsection carry over and we have only the examples in the statement of the Lemma.

So the picture is:



Assume γ has a non-zero label (in the marking for the Y-high weight of V). Then $V_\gamma(Q_Y)$ has an L'_Y-high weight given by the labelling on the boxed nodes above with a 1 on the node to the right of γ and the 1 on the end node as pictured. The dimension of this D_{10}-module is at least the number of conjugates of the high weight, which is $2^9 \cdot 10$. But then $\dim(V^2(Q_Y)) \ge \dim(V_\gamma(Q_Y)) \ge 2^9\cdot 10 > 6\cdot 2^9 = 6\dim(V^1(Q_Y)) = \dim(Q^\alpha)\dim(V^1(Q_Y))$, which is a contradiction to Lemma 2.9. So γ has label 0. By Lemma 4.3, all the other nodes in the diagram for Y have label 0, so in fact V is a spin representation of Y.

Now switch parabolics: Let P_X correspond to $\{\beta_1,\beta_2\} < \Pi(X)$. As always, embed P_X in a parabolic P_Y of Y, via the usual construction (given in Lemma 2.7). Since P_X is t-stable, we again have in this case $[V,Q_Y]=[V,Q_X]$ by Lemma 2.8. Since $V/[V,Q_X] \neq 0$, the subset of $\Pi(Y)$ to which P_Y corresponds must contain the node α that has a label 1. Let L_i be the simple factor of L_Y that contains $\langle U_\alpha \rangle$.

The irreducible L'_X-module $V_1/[V_1,Q_X]$ has high weight $(e\delta_1+\delta_2)|_{T'_{L_X}}$. Its dimension is $(e+1)(e+3)$ if $e \neq p-2$, and $\frac{(e+1)(e+6)}{2}$ if $e=p-2$. Recall that $\dim(V/[V,Q_Y]) = 2(\dim(V_1/[V_1,Q_X]))$.

If L_i has type D_l, then $\dim(V_1/[V_1,Q_X])$ is a power of 2. The dimension $(e+1)(e+3)$ is a power of 2 only when $e=1$, and $\frac{(e+1)(e+6)}{2}$ is never a power of 2. So the possibility here is $e=1$, $p \neq 3$.

If L_i has type A_l and $e > 0$, then $l \geq \frac{(e+1)(e+6)}{2} - 1$ (which is always ≥ 6, so $\Pi(L_i)$ does not contain both α_n and α_{n-1}). Then $V^2(Q_Y)$ has a composition factor of high weight $(\lambda - \gamma_l - \gamma_{l-1} - \alpha)|_{T_{L_Y}}$ (where γ_{l-1}, γ_l are the end nodes of L_i, and α the node at the end of Y which is left out of L_i), of dimension $\binom{l+1}{3}$. So we have $\binom{l+1}{3} \leq 6 \dim(V^1(Q_Y)) = 6(l+1)$ (using Lemma 2.9 again). But $\binom{l+1}{3} \leq 6(l+1)$ is a contradiction for $l \geq 6$, and $l \leq 5$ only for $e = 0$.

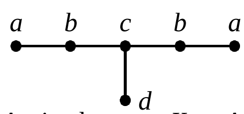
We are left with some cases for $e = 1$ and $e = 0$. But V is a spin module for Y, so $\dim(V)$ is a power of 2; the only time when V_1 as above has dimension a power of 2 is for $e = 1$ and $p > 11$ or $p = 0$, in which case $\dim(V_1) = 2^{15}$. But this would imply that $\dim(V) = 2^{16}$, so Y has type D_{17}. But $X = D_4$ has no irreducible representations of dimension 34 ($= \dim(W)$) when $p > 11$ or $p = 0$. □

This completes the proof of the Theorem. □

5. THE CASE $X = E_6$

Here we establish Theorem 1 for the case where $X = E_6$ and $G = X\langle t \rangle$. Again we assume that t acts on W, the natural module for Y. Notation ($X \leq Y$, V, V_1, V_2, α_i, λ) is as previously; in particular, $\{\beta_1, \dots, \beta_6\}$ is the set of simple roots of X, with $\{\delta_1, \dots, \delta_6\}$ the set of fundamental dominant weights, labelled so that $\langle \delta_i, \beta_j \rangle = \delta_{ij}$. The main theorem is:

Theorem 5.1. *If X acts irreducibly on the natural module W for Y then X is not of type E_6.*

Proof. Since t acts on W, the T_X-high weight $\delta = d_1\delta_1 + \cdots + d_6\delta_6$ of W must be symmetric; that is, $d_1 = d_6$ and $d_3 = d_5$. We will write $\delta = a\delta_1 + d\delta_2 + b\delta_3 + c\delta_4 + b\delta_5 + a\delta_6$, so the picture of the X-high weight of W is: 

. We use the same method as in section 3, based on Lemma 3.2. As in the case $X = A_m$, we will investigate the embedding of the fixed Borel subgroup B_X of X in a parabolic subgroup P_Y of Y, via the construction outlined in the proof of Lemma 2.7. In the first subsection, we will show that there are only a few cases in which a factor of type A_1 might appear in L'_Y; following that, we deal with these few cases by investigating the embeddings of other parabolic subgroups of X.

5.1. **The Almost-Everywhere Argument.** The argument will again be an induction on the partial order on the weight lattice; for the base case of the induction we need:

Lemma 5.2. *If X is of type E_6 and $\delta \neq \delta_2$, then $\delta \succ \delta_1 + \delta_6$.*

Proof. This is an easy exercise (using the fact that δ is symmetric). □

Notice that 0 is a weight of $V_{E_6}(\delta_1 + \delta_6)$ at level 16 by the expression for $\delta_1 + \delta_6$ in terms of roots (see [5], for example). So to begin our induction we must show that in $V_{E_6}(\delta_1 + \delta_6) = V_{E_6}(\delta)$ there are at least three weights in every U_X-level i for $2 \leq i < 16$, and at least 5 at level 16. This is easy to do; we illustrate by giving three weights in each of levels 2-4. Here we are using the usual rules to determine that a weight appears: If μ is a T_X-weight such that $W_\mu \neq 0$, and $\langle \mu, \beta_i \rangle = l > 0$, then $W_{\mu - j\beta_i} \neq 0$ for every $1 \leq j \leq l$. This depends on the result in [8], which says that weights which appear in characteristic 0 also appear in characteristic p, and the fact that in characteristic 0, the β_i-string through μ is connected (the set of weights is saturated — see [5, p. 114]).

Level 2: $\delta-\beta_1-\beta_3$, $\delta-\beta_6-\beta_5$, $\delta-\beta_1-\beta_6$;
Level 3: $\delta-\beta_1-\beta_3-\beta_4$, $\delta-\beta_6-\beta_5-\beta_4$, $\delta-\beta_1-\beta_3-\beta_6$;
Level 4: $\delta-\beta_1-\beta_3-\beta_4-\beta_5$, $\delta-\beta_6-\beta_5-\beta_4-\beta_3$, $\delta-\beta_1-\beta_3-\beta_4-\beta_6$;
etc.

If $\delta=\delta_2$, then we check all levels and find that the only possibilities for an A_1-factor of L_Y' are level 3 and level 11 (the level of the 0 weight). The weight space for the weight 0 has dimension at least 5 in all characteristics, so level 11 is large enough to preclude a corresponding a_1-factor of L_Y'. We will deal with level 3 later.

Since δ is symmetric, it has the form $a(\delta_1+\delta_6)+b(\delta_3+\delta_5)+c\delta_4+d\delta_2$ for some nonnegative integers a,b,c,d. Now for the induction: Assume $\delta\succ\delta_1+\delta_6$ (and $\delta\neq\delta_1+\delta_6$). Then at least one of the following must be true: 1) $d\geq 2$; 2) $c\geq 2$; 3) $b>0$; 4) $a>0$; or 5) $\delta\in\{\delta_4,\delta_2+\delta_4\}$. We consider each of these possibilities in turn.

1) If $d\geq 2$, then $\delta-\beta_2$ is a dominant weight, still greater than $\delta_1+\delta_6$ in the partial order, and by induction δ has enough weights at all levels 3 and higher. So we need to check levels 1 and 2. Level one is 2-dimensional if $a=b=0, c\neq 0$; otherwise $\dim(W_1)=1$ or $\dim(W_1)\geq 3$. At level 2, if $a=b=c=0$, we have just the two weights $\delta-2\beta_2$, $\delta-\beta_2-\beta_4$; so this is a case we must consider below.

2) If $c>1$, then $\delta-\beta_4$ is a dominant weight, still greater than $\delta_1+\delta_6$ in the partial order, and by induction δ has enough weights at all levels 3 and higher. So we need to check levels 1 and 2. As above, level one is 2-dimensional if $a=b=0, d\neq 0$. At level two are the three weights $\delta-\beta_4-\beta_3$, $\delta-\beta_4-\beta_2$, and $\delta-\beta_4-\beta_5$.

3) If $b>0$, then $\delta-\beta_3-\beta_4-\beta_5$ is a dominant weight, still greater than $\delta_1+\delta_6$, and by induction we must check levels 1-4. Level one is 2-dimensional only if $a=c=d=0$; level 2 has at least 5 weights in it; level 3 at least 5; and level 4 at least 5. So for $b\neq 0$ we need to consider only level 1 with $a=c=d=0$.

4) If $a>0$ and $\delta\neq\delta_1+\delta_6$, then $\delta-\beta_1-\beta_3-\beta_4-\beta_5-\beta_6$ is a dominant weight, and we need to check levels 1-6. Level one is 2-dimensional only if $b=c=d=0$; level 2 has at least 5 weights appearing; level 3 at least 4; level 4 at least 5; level 5 at least 5; and level 6 at least 5. So for $a\neq 0$, we have only to consider level 1 with $b=c=d=0$.

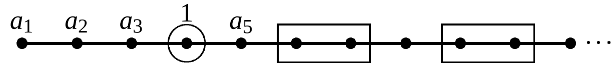
5) The weights that aren't covered above are: $\delta=\delta_4$, $\delta=\delta_2+\delta_4$, and $\delta=\delta_2$. Notice that $\delta_4-\beta_3-\beta_2-2\beta_4-\beta_5=\delta_1+\delta_6$, so we need to check levels 1-6; in this case, level 1 has dimension 1, and levels 2–6 all have dimension at least 3. So $\delta=\delta_4$ gives no A_1-factors of L_Y'. If $\delta=\delta_2+\delta_4$, then $\delta-\beta_4-\beta_2$ is a dominant weight, so we need to check only levels 1-3. Level one has dimension 2, so must be considered; levels 2 and 3 are both big enough.

So the only embeddings for which an A_1 factor of L_Y' might appear are the ones that give an obvious level of dimension 2:

1. Level 1 for $\delta=a\delta_1+a\delta_6$; $\delta=b\delta_3+b\delta_5$; or $\delta=c\delta_4+d\delta_2$ ($a\neq 0\neq b$, $c\neq 0\neq d$).
2. Level 2 for $\delta=d\delta_2$ with $d>1$.
3. Level 3 for $\delta=\delta_2$.

5.2. **The Remaining Cases.** We treat the cases listed above in reverse order:

3. Say $\delta=\delta_2$. Then the picture of the embedding of the Borel subgroup of X is



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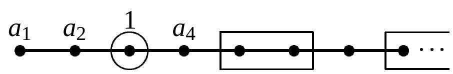
with all simple factors of L_Y' to the right of the picture having rank at least 2, all separated by only a single node.

By Lemma 2.4, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) = 2\dim(V_1^2(Q_X)) \leq 2\cdot 6\dim(V_1^1(Q_X)) = 12$. No node other than α_4 of the Dynkin diagram for Y which falls in $\Pi(L_Y')$ can have a nonzero label in the labelling for λ on Y, since $\dim(V^1(Q_Y)) = 2$. Suppose some node $\alpha_i \in \Pi(Y) - \Pi(L_Y)'$ for $i > 7$ has a nonzero label. Then $(\lambda - \alpha_i)|_{T_{L_Y'}}$ is a high weight in $V^2(Q_Y)$, giving a composition factor of dimension at least 9 (since the two simple factors of L_Y' which it adjoins have rank ≥ 2). But $\lambda - \alpha_4 - \alpha_3$ (if $a_3 = 0$) or $\lambda - \alpha_3$ (if $a_3 > 0$) is another high weight in $V^2(Q_Y)$, giving a factor of dimension at least 1; and $\lambda - \alpha_4 - \alpha_5$ (if $a_5 = 0$) or $\lambda - \alpha_5$ (if $a_5 > 0$) is another, giving one of dimension at least 3. Adding these up, we see that $\dim(V^2(Q_Y)) \geq 13$, which is a contradiction.

So $a_i = 0$ for $i > 5$. Now embed the maximal parabolic subgroup P_X of X corresponding to β_2 into a parabolic subgroup P_Y of Y by the same Q_X-level construction. Here level 0 has dimension 1 and level 1 has dimension 20. Now L_X' has type A_5 and the factor L_1 of L_Y', corresponding to level 1, has type A_{19}. Since $a_i = 0$ for $i > 5$, L_1 is the only factor of L_Y' acting nontrivially on $V/[V,Q_Y]$. By Lemma 2.8, $V/[V,Q_Y] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X]$. But then we are in the situation of section 3, with $V/[V,Q_Y]$ an irreducible $A_5\langle t\rangle$-module. Since there were no examples there of this setup, inductively we have none here.

2. If $\delta = d\delta_2$ with $d > 1$, then the embedding of the Borel subgroup of X in a parabolic subgroup of Y is



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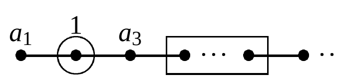
with, as above, all simple factors of L_Y' to the right of the picture of rank at least 2, all separated by only a single node. We continue exactly as in case 3, finding no examples.

1. We have $\delta = a\delta_1 + a\delta_6$, $\delta = b\delta_3 + b\delta_5$, or $\delta = c\delta_4 + d\delta_2$. The embedding of B_X in a parabolic subgroup of Y is as pictured in



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with the second simple factor of L_Y' having type A_l for $l \geq 2$ and all simple factors of L_Y' to the right of the picture having rank at least 2, all separated by only a single node. By the same arguments as above, $a_i = 0$ for $i > 4$.

We again embed the maximal parabolic subgroup of X corresponding to β_2 in a parabolic subgroup of Y; now level 0 is large, with dimension at least 20. Because $a_i = 0$ for $i > 4$, the factor L_1 of L_Y' corresponding to this level 0 is the only simple factor of L_Y' acting nontrivially on $V/[V,Q_Y]$. As above, we are back in the situation of section 3, which tells us that once again there are no examples.

So $G = E_6\langle t\rangle$, with E_6 acting irreducibly on W, gives no examples. □

6. THE CASE $X = D_4$, $[G : X] \geq 3$

In this section we treat the cases $G = D_4\langle s\rangle$ and $G = D_4\langle s,t\rangle$, where s is a graph automorphism of $D_4 = X$ of order 3 and t is one of order 2. As mentioned earlier, since $G < \mathrm{Aut}(Y)$ we have $s \in Y$, as no simple algebraic group properly containing D_4 has an outer automorphism of order 3. We assume that s acts on W and that W is irreducible as an X-module. The case $G = D_4\langle t\rangle$ was covered in section 4. All notation is as before. The main result is:

Theorem 6.1. *If X acts irreducibly on the natural module W for Y, then we are not in the situation where $X = D_4$ with $G = X\langle s\rangle$ or $G = X\langle s,t\rangle$.*

Notice that the assumption that s acts on W forces the T_X-high weight δ of W to be of the form $a\delta_1+b\delta_2+a\delta_3+a\delta_4$, which implies that t acts on W. In addition, it implies that X fixes a nondegenerate bilinear form on W, which is orthogonal if $p\neq 2$ ([11, Lemma 79]).

If $V|_X=V_1\oplus V_2\oplus V_3$ with each V_i irreducible and restricted as an $X=D_4$-module, then s permutes the V_i and V is irreducible as an $X\langle s\rangle$-module; in this case we may assume that $G=X\langle s\rangle$. So if $G=X\langle s,t\rangle$, we may assume that V has six simple factors as an X-module.

We based the arguments for $X=A_m$ and $X=E_6$ on Lemma 3.2. We will need an analogous result for the cases we consider here. Recall that l_δ is the U_X-level of the T_X-low weight of W. Let P_Y be the parabolic subgroup of Y containing $B_X=U_XT_X$, constructed via U_X-levels as outlined in Lemma 2.7, with $P_Y=Q_YL_Y$ the Levi decomposition given in that construction.

Lemma 6.2. *If P_Y is as above, then $\dim(V/[V,Q_Y])=3$ if $G=X\langle s\rangle$, and $\dim(V/[V,Q_Y])=6$ if $G=X\langle s,t\rangle$. If $G=X\langle s\rangle$, either $\dim(W_{l_\delta/2})=4$ or there is U_X-level in W of dimension 3 or 2. If $G=X\langle s,t\rangle$, there is an i such that $2\leq\dim(W_i)\leq 6$.*

Proof. It suffices to prove the statements in the first sentence, since A_1, A_2, and $A_1\times A_1$ are the only groups under consideration which have simple modules of dimension 3, and only those groups whose natural modules have dimension at most 6 have an irreducible module of dimension 6.

The construction via U_X-levels of the parabolic subgroup P_Y of Y clearly gives an s-stable subgroup, as s acts on W and on each U_X-level in W. The quotient $V/[V,Q_Y]$ is an irreducible L_Y'-module by [9, Lemma 2.10]. By an argument similar to that used to prove Lemma 2.8, here we have $(V/[V,Q_Y])|_{L_X}=V_1/[V_1,U_X]\oplus V_2/[V_2,U_X]\oplus V_3/[V_3,U_X]$ $(\oplus V_4/[V_4,U_X]\oplus V_5/[V_5,U_X]\oplus V_6/[V_6,U_X]$, if $t\in G)$. But each of these L_X-modules $V_i/[V_i,U_X]$ has dimension 1 by Lemma 2.4. This proves the first statement of the Lemma.

Recall that P_Y is the stabilizer in Y of the flag $0\leq W_l\leq W_l\oplus W_{l-1}\leq\cdots$, where l is minimal with respect to $[W,U_X^{l+1}]=0$. Each factor L_i of L_Y' corresponds to a U_X-level W_i. Since L_Y' has an irreducible module of dimension $j=3$ or 6, there must be a simple factor of L_Y' of rank less than j. For $j=3$, this can happen if $i=l_\delta/2$ and $\dim(W_{l_\delta/2})=4$ (we could have the middle level $l_\delta/2$ giving a product of two groups of type A_1 if $Y=D_n$); otherwise, this happens only if $\dim(W_i)\leq j$. This proves the second assertion of the Lemma. □

Now if we can show that for a particular T_X-high weight δ of W, all U_X-levels of W have dimension bigger than j (with j as above), and in the $G=X\langle s\rangle$ case $\dim(W_{l_\delta/2})>4$, then the Lemma will imply that there are no examples with this embedding of X into Y.

Lemma 6.3. *Assume $X=D_4$, $s\in G$, and X acts irreducibly on the natural module W for Y. If $\dim(W_i)\leq 3$ where W_i is the U_X-level i of W, then one of the following holds:*

1. *$i=1$ and either $\delta=b\delta_2$ or $\delta=a\delta_1+a\delta_3+a\delta_4$ for some $b\neq 0\neq a$.*
2. *$\delta=\delta_2$, $\delta=2\delta_2$, or $\delta=\delta_1+\delta_3+\delta_4$.*

   If $\dim(W_i)\leq 6$ then one of the following holds:
3. *$i=1$.*
4. *$i=2$ and either $\delta=b\delta_2$ for some $b\neq 0$ or $\delta=\delta_1+\delta_2+\delta_3+\delta_4$.*
5. *$\delta=\delta_2$, $\delta=2\delta_2$, or $\delta=\delta_1+\delta_3+\delta_4$.*

Finally, if $\dim(W_{l_\delta/2})\leq 4$, then $\delta=\delta_2$.

Proof. We wish to induct on the height in the weight lattice of δ, as in the $X=A_m$ and $X=E_6$ cases. Let

$$\delta=a\delta_1+b\delta_2+a\delta_3+a\delta_4$$

be the T_X-high weight of W. Since $\delta_1+\delta_3+\delta_4 = 2\beta_1+3\beta_2+2\beta_3+2\beta_4$ and $\delta_2 = \beta_1+2\beta_2+\beta_3+\beta_4$ are sums of roots, every weight which has the form of δ has $\delta_1+\delta_2+\delta_3+\delta_4$ as a subdominant weight, except δ_2, $2\delta_2$, and $\delta_1+\delta_3+\delta_4$.

So to begin our induction we must investigate the numbers of weights at various U_X-levels of the D_4-module with high weight $\delta_1+\delta_2+\delta_3+\delta_4$. The weight 0 appears in this module at level 14, so we must check the numbers of weights at levels i with $1 \le i \le 14$. It is not hard to do this (again using the result in [8] that weights which appear in characteristic 0 appear in characteristic p); we find that there are 4 weights at level 1, 6 at level 2, and 10 or more at every level 3-14. We will exclude level 1 from the discussion below, since it is clear that $\dim(W_1) = 1$ if $a = 0$ and $b > 0$; $\dim(W_1) = 3$ if $a > 0$ and $b = 0$; and $\dim(W_1) = 4$ if $a \neq 0 \neq b$.

Assume $\delta = a\delta_1+b\delta_2+a\delta_3+a\delta_4$ as above, with $b > 2$. Then $\delta-\beta_2$ is a dominant weight, less than δ in the partial order and still having $\delta_1+\delta_2+\delta_3+\delta_4$ as a subdominant weight, so by induction $\delta-\beta_2$ has enough weights at all levels 3 and higher. So we must check δ-levels 2 and 3. There are 4 weights at level 2 if $a = 0$, and 7 otherwise. There are at least 7 weights at level 3. So the only possibility for a level of dimension 6 or less is level 2, with $a = 0$.

If $b = 2$ and $a > 1$, then again $\delta-\beta_2$ is dominant and lower in the partial order, still with $\delta_1+\delta_2+\delta_3+\delta_4$ as a subdominant weight, so by induction $\delta-\beta_2$ has enough weights at all levels 3 and higher. Here, level 2 has 7 weights and level 3 has at least 7. So we have no possibilities arising from this embedding (other than level 1).

Finally, if $b = 1$ and $a > 1$, then $\delta-\beta_1-\beta_3-\beta_4$ is dominant and lower in the partial order; so by induction we must check δ-levels 2, 3, 4, and 5. At all four of these levels there are more than seven weights. This completes the listing of all possibilities for levels of dimension at most than 6, except for the three weights δ_2, $2\delta_2$, and $\delta_1+\delta_2+\delta_3$.

The last statement of the Lemma is proved by noting that by the same argument as above, every symmetric dominant weight except δ_1 has $\delta_1+\delta_3+\delta_4$ as a subdominant weight, and $\delta_1+\delta_3+\delta_4$ has more than 4 weights at level 9, which is its middle level. □

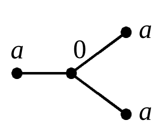
We see in the proof that the only possibilities for 1-dimensional U_X-levels are level 1 when $\delta = b\delta_2$, and level 0. So as in the arguments for earlier cases, we have that for any δ, there are never 2 consecutive weights in the Dynkin diagram for Y which lie outside of $\Pi(L'_Y)$, except for α_1 and α_2 in the case $\delta = b\delta_2$.

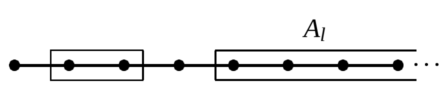
From this point, we will consider the two possibilities for G separately.

6.1. $G = X\langle s\rangle$**.** Here we need to find a level of dimension 3 or 2. By the previous lemma, we need to consider only level 1 for $\delta = a\delta_1+a\delta_3+a\delta_4$, and the three possible δ that were not covered by the induction there: δ_2, $2\delta_2$, and $\delta_1+\delta_3+\delta_4$. We can check these last three directly and find that the only possibilities for a factor of L'_Y of type A_1 or A_2 are those corresponding to level 1 for $\delta = \delta_1+\delta_3+\delta_4$, and levels 2, 3 and 5 for $\delta = \delta_2$.

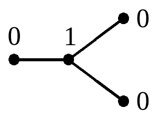
So we must consider the following:

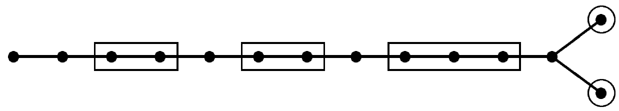
$W|_X$ Embedding of B_X

1) 

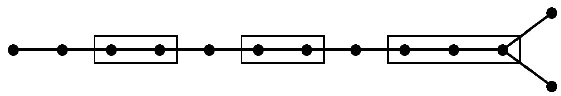


with $l = \begin{cases} 8 & \text{if } a > 1, \\ 5 & \text{if } a = 1. \end{cases}$

2) 



If $p \neq 2$



If $p = 2$

1) Assume $\delta = a\delta_1 + a\delta_3 + a\delta_4$. By the last Lemma, $\dim(V/[V,Q_Y]) = 3$, so either $a_2 = 1$ or $a_3 = 1$, with all other $a_i = 0$ for $\alpha_i \in \Pi(L_Y')$. By Lemma 2.4, $\dim(V^2(Q_Y)) \leq \dim(V^2(U_X)) = 3\dim(V_1^2(U_X)) \leq 3(4\dim(V_1^1(U_X))) = 12$. If $a_3 = 1$, then in $V^2(Q_Y)$ we have either $\lambda - \alpha_4$ (if $a_4 \neq 0$) or $\lambda - \alpha_3 - \alpha_4$ (if $a_4 = 0$) as a high weight in $V^2(Q_Y)$. The L_Y'-modules with these high weights have dimension greater than 12, however, so in fact $a_3 = 0$.

If $a_2 = 1$, then similar calculations give contradictions to $\dim(V^2(Q_Y)) \leq 12$ if $0 \neq a_1 \neq p-2$ or $a_i \neq 0$ for some $i > 2$, or if $a > 1$ and $a_1 \neq 0$. So the only possibilities here are $a_2 = 1$ with $a_i = 0$ for $i \neq 2$ (i.e. $\lambda = \lambda_2$), and $a = 1$, $a_1 = p-2$, $a_2 = 1$ with $a_i = 0$ for $i > 2$ (so $\lambda = (p-2)\lambda_1 + \lambda_2$).

Assume $p = 2$. Then, since δ is restricted by Lemma 2.6, $\delta = \delta_1 + \delta_3 + \delta_4$. In characteristic 2, the D_4-module with this high weight has dimension 294, so Y has type C_{147} or D_{147}. By the previous paragraph, we have $a_1 = 0$. But then $\lambda = \lambda_2$ has $(2^{147} \cdot 147!)/(2 \cdot 2^{145} \cdot 145!) = 2 \cdot 147 \cdot 146 = 42,924$ conjugates. So $\dim(V) \geq 42,924$. The T_X-high weights of V are restricted as well and not symmetric with respect to s, so the possibilities for these are very limited. In fact, any D_4-module with such a high weight has dimension at most 840; but then $42,924 \leq \dim(V) = 3\dim(V_1) \leq 3 \cdot 840$ gives a contradiction. So we may assume $p \neq 2$.

Assume $\lambda = \lambda_2$; this implies $V \cong \bigwedge^2 W$ ([7, II.2.15]). If $w \in W_\delta$, $w_1 \in W_{\delta-\beta_1}$, $w_2 \in W_{\delta-\beta_3}$, and $w_3 \in W_{\delta-\beta_4}$, then $w \wedge w_1$, $w \wedge w_2$, and $w \wedge w_3$ are T_X-high weight vectors in $\bigwedge^2 W$, of weights $(2a-2)\delta_1 + \delta_2 + 2a\delta_3 + 2a\delta_4$, $2a\delta_1 + \delta_2 + (2a-2)\delta_3 + 2a\delta_4$, and $2a\delta_1 + \delta_2 + 2a\delta_3 + (2a-2)\delta_4$. Since we are assuming there are only three such T_X-high weights, V must be the sum (as X-modules) of the three D_4-modules with these high weights.

Now switch parabolics, and embed the parabolic subgroup P_X of X corresponding to $\{\beta_1, \beta_3, \beta_4\}$ in a parabolic subgroup P_Y of Y via the Q_X-level construction. Then the factor L_1 of L_Y' corresponding to level 0 is the only simple factor to act nontrivially on $V/[V,Q_Y]$, and is of type A_l with $l = (a+1)^3 - 1$. Now $V/[V,Q_Y] \cong_{L_Y'} \bigwedge^2(W/[W,Q_Y]) = \bigwedge^2$(the natural module for L_1), and $(V/[V,Q_Y])|_X$ has the three high weights $(2a-2, 2a, 2a)$, $(2a, 2a-2, 2a)$, and $(2a, 2a, 2a-2)$. But this gives

$$\binom{(a+1)^3}{2} = \dim\left(\textstyle\bigwedge^2(W/[W,Q_Y])\right) = 3\dim(V_1/[V_1,Q_X]) = 3((2a-1)(2a+1)^2),$$

and this equation in a has no solutions in the positive integers. So $\lambda \neq \lambda_2$.

Finally, assume $a = 1$ and $\lambda = (p-2)\lambda_1 + \lambda_2$. Then $\lambda_1|_{T_X} = \delta_1 + \delta_3 + \delta_4$ and as above we may assume that $\lambda_2|_{T_X} = (2\lambda_1 - \alpha_1)|_{T_X} = 2(\delta_1 + \delta_3 + \delta_4) - \beta_i$ for some $i = 1, 3$, or 4. So $\lambda_2|_{T_X}$ is one of $\delta_2 + 2\delta_3 + 2\delta_4$, $2\delta_1 + \delta_2 + 2\delta_4$, or $2\delta_1 + \delta_2 + 2\delta_3$. In any case, $\lambda|_{T_X} = ((p-2)\lambda_1 + \lambda_2)_{T_X} = (p-2)\lambda_1|_{T_X} + \lambda_2|_{T_X}$ is a T_X-high weight of V which is not restricted, contrary to our assumption. This eliminates the final possibility for case 1.

2) If $\delta = \delta_2$, then P_Y corresponds to the indicated nodes of the Dynkin diagram for Y, and the possibilities are that there is a label of 1 on one of the nodes in an A_2 factor of L'_Y, or a label of 2 on one of the nodes corresponding to an A_1 factor. We have $\dim(V^2(Q_Y)) \leq 12$ by Lemma 2.4, and the same sorts of arguments as above give contradictions to this bound for any of these possibilities except $a_4 = 1$ and $a_3 = 1$; even further, we get contradictions to the bound unless a) $\lambda = a_1\lambda_1 + a_2\lambda_2 + \lambda_3$, or b) $\lambda = \lambda_4$.

a) Suppose $\delta = \delta_2$ and $\lambda = a_1\lambda_1 + a_2\lambda_2 + \lambda_3$. Then $\lambda_1|_{T_X} = \delta_2$, $\lambda_2|_{T_X} = (2\lambda_1 - \alpha_1)_{T_X} = 2\delta_2 - \beta_2 = \delta_1 + \delta_3 + \delta_4$, and $\lambda_3|_{T_X} = (3\lambda_1 - 2\alpha_1 - \alpha_3)_{T_X} \in \{2\delta_1 + 2\delta_3, 2\delta_3 + 2\delta_4, 2\delta_1 + 2\delta_4\}$. So λ restricts to T_X as $a_2\delta_1 + a_1\delta_2 + (a_2+2)\delta_3 + (a_2+2)\delta_4$ or one of its s-conjugates.

If we now embed the parabolic subgroup P_X of X corresponding to $\{\beta_1, \beta_3, \beta_4\} \subseteq \Pi(X)$ in a parabolic subgroup $P_Y = Q_Y L_Y$ of Y, then the first nontrivial factor L_1 of L'_Y corresponds to $\{\alpha_2, \alpha_3, ..., \alpha_8\} \subseteq \Pi(Y)$ (since Q_X-level 1 has dimension 8). Here L'_X is a product of three groups of type A_1, and $(V/[V,Q_Y])_{L'_X}$ is the sum of three simple L'_X-modules, with high weights (a_2, a_2+2, a_2+2), (a_2+2, a_2, a_2+2), and (a_2+2, a_2+2, a_2). So $\dim(V^1(Q_Y)) = 3(a_2+1)(a_2+3)^2$. On the other hand, $V^1(Q_Y)$ is isomorphic as an L_1-module to the A_7-module M with high weight $\gamma = a_2\gamma_1 + \gamma_2$, where the γ_i are the fundamental dominant weights for A_7. Assume that $a_2 > 15$. Then γ has $(a_2-6)\gamma_1$ as a subdominant weight, and the A_7-module with high weight $(a_2-6)\gamma_1$ has $\binom{a_2+1}{a_2-6}$ nonzero weights by Lemma 2.5. So $\dim(V^1(Q_Y)) \geq \binom{a_2+1}{a_2-6}$, and $\binom{a_2+1}{a_2-6} > 3(a_2+1)(a_2+3)^2$ for $a_2 > 15$, which is a contradiction.

If $a_2 \leq 15$, we can check each case individually, finding more than $3(a_2+1)(a_2+3)^2$ weights in M, arriving at the same contradiction. So (a) is ruled out.

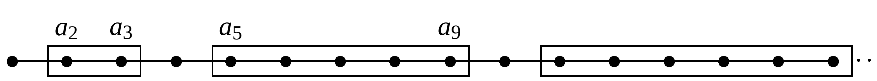
b) Suppose $\delta = \delta_2$ and $\lambda = \lambda_4$. As above, we have $\lambda_4|_{T_X} = (4\lambda_1 - 3\alpha_1 - 2\alpha_2 - \alpha_3)_{T_X} \in \{4\delta_2 - 3\beta_2 - \beta_j - \beta_k | j,k = 1,3 \text{ or } 4, j \neq k\} = \{3\delta_1 + \delta_3 + \delta_4, \delta_1 + 3\delta_3 + \delta_4, \delta_1 + \delta_3 + 3\delta_4\}$. When we embed the parabolic subgroup P_X of X as above, we now have $\dim(V^1(Q_Y)|_{L'_X}) = 3(2 \cdot 2 \cdot 4) = 48$ on the one hand, and $\dim(V^1(Q_Y)) = \binom{8}{3} = 56$ on the other (since $V^1(Q_Y)$ is isomorphic to the A_7-module with high weight γ_3). Again we have a contradiction, and (b) is eliminated.

So there are no examples with $G = D_4\langle s \rangle$.

6.2. $G = X\langle s,t \rangle$**.** Assume $X = D_4$ with $G = X\langle s,t \rangle$. Lemma 6.2 tells us that we must have a U_X-level in W of dimension 6 or less. Lemma 6.3 shows that there are few T_X-high weights δ of W which allow such a level (other than level 1). We must consider level 1 for all δ, level 2 for $\delta = b\delta_2$ or $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$, and the three possible δ not covered by the argument in the proof of that Lemma (δ_2, $2\delta_2$, and $\delta_1 + \delta_3 + \delta_4$).

As above, by Lemma 2.4 we have here $\dim(V^2(Q_Y)) \leq 4\dim(V^1(Q_Y)) = 24$. This rules out most of the remaining possibilities.

Assume $\delta = \delta_1 + \delta_3 + \delta_4$. Then we check the dimensions of the levels directly. With W_i denoting U_X-level i of W, we have $\dim(W_0) = 1$, $\dim(W_1) = 3$, $\dim(W_2) = 6$, and $\dim(W_i) \geq 7$ for $i \geq 3$. So P_Y corresponds to the indicated nodes in the Dynkin diagram for Y in this picture:



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Lemma 6.2 tells us that $\dim(V^1(Q_Y)) = 6$; the ways in which this can happen given the dimensions of U_X-levels above are a) $a_2 = 2$ and $a_i = 0$ for $i \neq 2$, $\alpha_i \in \Pi(Y) - \Pi(L'_Y)$; b) $a_3 = 2$ and $a_i = 0$ for $i \neq 3$, $\alpha_i \in \Pi(Y) - \Pi(L'_Y)$; c) $a_5 = 1$ and $a_i = 0$ for $i \neq 5$, $\alpha_i \in \Pi(Y) - \Pi(L'_Y)$; and d) $a_9 = 1$ and $a_i = 0$ for $i \neq 9$, $\alpha_i \in \Pi(Y) - \Pi(L'_Y)$. We consider each possibility in turn (the arguments below must be modified slightly in some small characteristics):

a) If $a_2 = 2$, then either $\lambda - \alpha_1$ (if $a_1 \neq 0$) or $\lambda - \alpha_2 - \alpha_1$ (if $a_1 = 0$) is a high weight in $V^2(Q_Y)$, giving a composition factor there of dimension at least 7, and either $\lambda - \alpha_4$ (if $a_4 \neq 0$) or $\lambda - \alpha_2 - \alpha_3 - \alpha_4$ (if $a_4 = 0$) is another, giving a factor of dimension at least 18. This contradicts $\dim(V^2(Q_Y)) \leq 24$. So this case does not arise.

b) If $a_3 = 2$, then $\lambda - \alpha_4$ (if $a_4 \neq 0$) or $\lambda - \alpha_3 - \alpha_4$ gives a composition factor in $V^2(Q_Y)$ of dimension at least 42; this again is a contradiction.

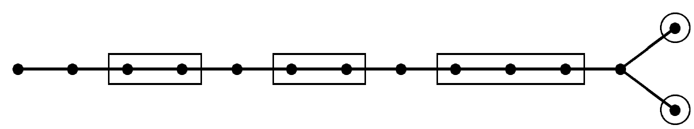
c) If $a_5 = 1$, then either $\lambda - \alpha_4$ (if $a_4 \neq 0$) or $\lambda - \alpha_4 - \alpha_5$ (if $a_4 = 0$) is a high weight in $V^2(Q_Y)$, giving $\dim(V^2(Q_Y)) \geq 45$. This again contradicts $\dim(V^2(Q_Y)) \leq 24$.

d) Similarly, here we obtain a contradiction since either $\lambda - \alpha_{10}$ or $\lambda - \alpha_9 - \alpha_{10}$ is a high weight in $V^2(Q_Y)$, giving $\dim(V^2(Q_Y)) \geq 105$.

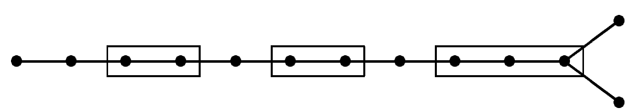
So the embedding $\delta = \delta_1 + \delta_3 + \delta_4$ gives no examples.

Almost identical arguments rule out all possibilities for $\delta = 2\delta_2$, so this embedding gives no examples.

For $\delta = \delta_2$, there are many more possibilities for the labelling of λ on $\Pi(L'_Y)$. Here the picture of P_Y is

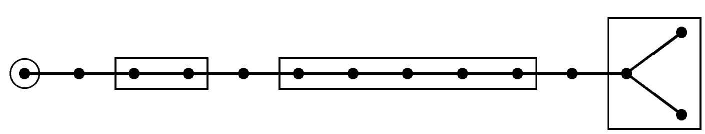


for $p \neq 2$, and



for $p = 2$ (in [9] it is calculated that this embedding is actually into D_{13}, not C_{13}). Once again, however, arguments like those above give contradictions to $\dim(V^2(Q_Y)) \leq 24$ for all possible λ except perhaps $\lambda = a_1\lambda_1 + a_2\lambda_2 + 2\lambda_3$, with $a_1 a_2 = 0$. So we may assume λ has this form, and $p \neq 2$ since λ has λ_3-coefficient 2.

If $a_1 = 0 \neq a_2$ and $\lambda = a_2\lambda_2 + 2\lambda_3$, then we use the s-stable minimal parabolic subgroup P_X of X corresponding to $\{\beta_2\} \subseteq \Pi(X)$. We embed P_X in a parabolic subgroup P_Y of Y corresponding to the indicated nodes in the picture



.

Now by Lemma 2.4, we have $\dim(V^2(Q_Y)) \leq 6\dim(V^1(Q_Y)) = 36$. But $\lambda - \alpha_2$ and $\lambda - \alpha_3 - \alpha_4 - \alpha_5$ are high weights in $V^2(Q_Y)$, giving composition factors of dimension 20 (if $p \neq 3$) and 18, respectively; this is a contradiction (if $p = 3$, there is another composition factor of high weight $\lambda - \alpha_2 - \alpha_3$ which we must include to obtain the contradiction).

If $a_2 = 0$ and $a_1 \neq 0$ (so $\lambda = a_1\lambda_1 + 2\lambda_3$), then we can examine the embedding of the s-stable parabolic subgroup of X corresponding to $\{\beta_1, \beta_3, \beta_4\} \subseteq \Pi(L'_Y)$ in a parabolic subgroup $P_Y = Q_Y L_Y$ of Y. Now the first nontrivial factor L_1 of L'_Y corresponds to $\{\alpha_2, \ldots, \alpha_8\}$

(as Q_X-level 1 has dimension 8), and the second factor L_2 corresponds to $\{\alpha_{10},...,\alpha_{14}\}$. The first, L_1, is the only one to act nontrivially on $V^1(Q_Y)$, which is isomorphic to the A_7-module with high weight $2\gamma_2$ (where γ_i are the fundamental dominant weights of A_7). We find (using the Weyl character formula and the Andersen-Jantzen sum formula) that $\dim(V^1(Q_Y)) = 336$ if $p \neq 3$, and $\dim(V^1(Q_Y)) = 266$ if $p = 3$. By Lemma 2.4, we have $\dim(V^2(Q_Y)) \leq 8\dim(V^1(Q_Y))$. With λ as above, the two high weights $\lambda-\alpha_1$ and $\lambda-\alpha_3-\cdots-\alpha_9$ in $V^2(Q_Y)$ give, for $p \neq 3$, two composition factors, each of dimension 1680; and for $p = 3$, these two composition factors have dimension 1624 and 1120, respectively. This contradicts $\dim(V^2(Q_Y)) \leq 8\dim(V^1(Q_Y))$.

The remaining possibility is $\lambda = 2\lambda_3$. We know that $\lambda_1|_{T_X} = \delta_2$, $\alpha_1|_{T_X} = \beta_2$ by [9, 3.4(i)], and $\alpha_2|_{T_X} = \beta_i$ for $i = 1,3$ or 4, by the same arguments used in the proof of [9, 3.4]. Since $\lambda_3 = 3\lambda_1 - 2\alpha_1 - \alpha_2$, this gives $\lambda_3|_{T_X}$ of the form $2\delta_i + 2\delta_j$ for $(i, j) = (1,3),(1,4)$, or $(3,4)$. But we have noted that the δ_1-, δ_3-, and δ_4-coefficients of $\lambda|_{T_X}$ must be distinct in this case $G = D_4\langle s,t\rangle$, since V is irreducible as a G-module. So the case $\delta = \delta_2$ does not occur.

Remaining to be considered are level 1 for all δ, and level 2 for $\delta = \delta_1+\delta_2+\delta_3+\delta_4$ or $\delta = b\delta_2$ (with $b > 1$, as the case $\delta = \delta_2$ was examined thoroughly above).

First we consider level 2. If $\delta = \delta_1+\delta_2+\delta_3+\delta_4$, then W_2 in fact has dimension 9 unless $p = 3$, in which case it has dimension 6 and we have the possibilities $a_6 \neq 0$ and $a_{10} \neq 0$. But these cases are easily ruled out by arguments based on $\dim(V^2(Q_Y)) \leq 24$, as in several cases above. If $\delta = b\delta_2$ then $\dim(W_2) = 4$ and we have the possibility $a_4 \neq 0$; this is again ruled out by the same sorts of arguments.

Finally, assume λ has a nonzero coefficient for a fundamental weight corresponding to some α_i in $\Pi(L_1)$, where L_1 is the simple factor of L_Y' corresponding to the U_X-level 1 of W (in other words, we are in the remaining level 1 case). Then if $a \neq 0 \neq b$, $\dim(W_1) = 4$ and we have the possibility $a_3 = 1$. If $a \neq 0 = b$, then $\dim(W_1) = 3$ and we have the possibilities $a_2 = 2$ and $a_3 = 2$. These all fall quickly to arguments based on the fact that $\dim(V^2(Q_Y)) \leq 24$ as above. If $b \neq 0 = a$, then $\dim(W_1) = 1$ and L_1 is trivial, so we do not have a possibility. This rules out level 1.

We have eliminated all possible high weights δ for W. This completes the proof of Theorem 6.1, which in turn completes the proof of Theorem 1. □

TABLE 1. Examples arising from the connected case

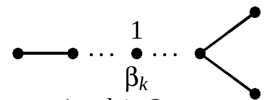
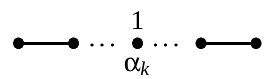
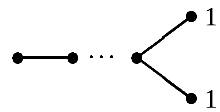
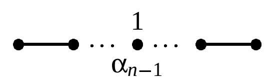
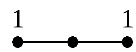
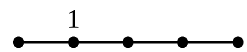
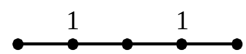
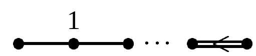
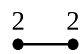
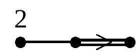
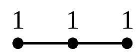
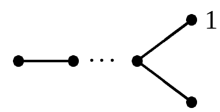
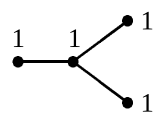
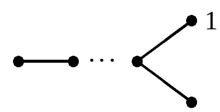
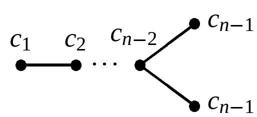
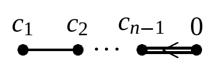
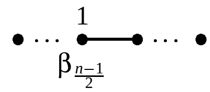
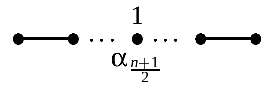
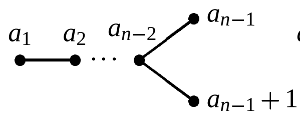
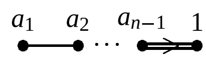
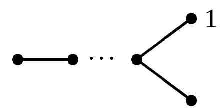
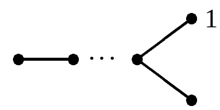
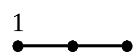
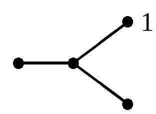
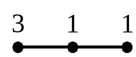
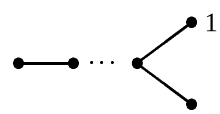
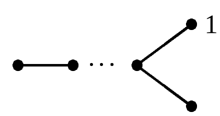
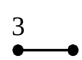
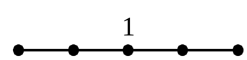
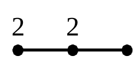
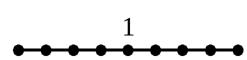
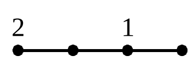
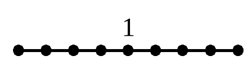
No.	X	Y	$W\|_X$	$V\|_X$	$V\|_Y$	char(K)
I_4	D_n $(n \geq 4)$	A_{2n-1}	δ_1	 $n-1 > k \geq 2$		$p \neq 2$
I_5	D_n $(n \geq 4)$	A_{2n-1}	δ_1			$p \neq 2$
I_6	A_3	A_5	δ_2			$p \neq 2$
II_1	A_5	C_{10}	δ_3			$p \neq 2$
S_1	A_2	B_3	$\delta_1 + \delta_2$			$p = 3$
S_7	A_3	D_7	$\delta_1 + \delta_3$			$p = 2$
S_8	D_4 $(s,t \in Y)$	D_{13}	δ_2			$p = 2$
MR_4	D_n	C_n	δ_1			$p = 2$

TABLE 2. New Examples

No.	X	Y	$W\|_X$	$V_1\|_X$	$V\|_Y$	char(K)
U_1	A_{n-1}	A_n (n odd)	usual			any
U_2	D_n	B_n	usual			below
			with $a_i + a_j \equiv i - j \pmod p$ whenever a_i and a_j are non-zero coefficients with only 0's between them and $i < j < n$; and $2a_i \equiv -2(n-i) - 1 \pmod p$ for a_i the last non-zero coefficient before $a_n = 1$.			
U_3	D_n	D_{n+1}	usual			any
U_4	A_3	D_4	$\delta_1 \oplus \delta_3$			any
U_5	A_3	D_{10}	$2\delta_2$			$p \neq 2,3,5,7$
U_6	D_m ($m > 3$)	C_m, B_m	δ_1			$p = 2$
U_7	A_2	A_5	$2\delta_1$			$p \neq 2,3$
U_8	A_3	A_9	$2\delta_1$			$p \neq 2,5$
U_9	A_4	A_9	δ_2			$p \neq 2,5$

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