OVERGROUPS OF IRREDUCIBLE LINEAR GROUPS, I

BEN FORD

1. INTRODUCTION

In work spread over several decades, Dynkin ([4, 3]), Seitz ([10, 11]), and Testerman ([16]) classified the maximal closed connected subgroups of simple algebraic groups. Their analyses for the classical group cases were based primarily on a striking result: If G is a simple algebraic group and $\varphi : G \to \mathrm{SL}(V)$ is a tensor indecomposable irreducible rational representation, then with specified exceptions the image of G is maximal among closed connected subgroups of one of the classical groups $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SO}(V)$. From a slightly different perspective, the question they answered was: Given an irreducible, closed, connected subgroup G of $\mathrm{SL}(V)$ for some vector space V, find all possibilities for closed, connected overgroups Y of G in $\mathrm{SL}(V)$.

This question of irreducible overgroups, or the restriction of irreducible modules to subgroups, appears in other contexts as well. In this paper we present some results in the absence of the connectedness requirement for the subgroup: The eventual goal is to classify all possible triples (G,Y,V) with $G < \mathrm{Aut}(Y)$ both closed irreducible subgroups of $\mathrm{SL}(V)$, $Y \neq \mathrm{SL}(V), \mathrm{SO}(V)$, or $\mathrm{Sp}(V)$, and Y a simple group of classical type. In this paper and [5], we give complete results for the case when G is not connected but has simple identity component X, and the T_Y-high weight and T_X-high weights of V are restricted. Specifically, the papers are concerned with the proof of Theorem 1 below.

Let G be a non-connected algebraic group with simple identity component X. Let V be an irreducible KG-module with restricted X-high weight(s).

Theorem 1. *Let Y be a simple algebraic group of classical type such that $X < Y < \mathrm{SL}(V)$, $G \leq \mathrm{Aut}(Y)$, and $V|_Y$ is irreducible with restricted high weight. Then $Y = \mathrm{SO}(V)$, $Y = \mathrm{Sp}(V)$, or (X,Y,V) appears in Table 1 or Table 2.*

If G has simple identity component X, then $G \leq \mathrm{Aut}(X)$. Since we require that $G \neq X$, we therefore may restrict our attention to X of type A_m, D_m, or E_6.

The group Y is of classical type and so has a natural module W. Some simply connected cover $\hat{Y}$ of Y acts irreducibly on W. Let $\tilde{X}$ be the preimage of $X < Y$ under the projection $\hat{Y} \to Y$. Then $\tilde{X} = \hat{X} \cdot \tilde{Z}$ for some cover $\hat{X}$ of X and some $\tilde{Z} \leq Z(\hat{Y})$. Now we replace Y by $\hat{Y}$ and X by $\hat{X}$. If u is an outer automorphism of X, then the action of u can be extended to $\hat{X}$; if $u \in Y$ let $\hat{u}$ be a preimage of u under the projection $\hat{Y} \to Y$. If u is an outer automorphism of Y, it is again possible to extend u to $\hat{u} \in \mathrm{Aut}(\hat{Y})$. Now replace u by $\hat{u}$. As we will make use only of the action of u on 1-spaces in Y- and X-modules, the possible extension by elements of $Z(\hat{Y})$ does not concern us. So we assume henceforth that Y is simply connected, and that X and Y act on W.

The analysis is different depending on whether X acts reducibly or irreducibly on W. We settle the reducible case in this paper, and the irreducible in [5]. Also, we will assume in [5] that the involutory graph automorphism of X, if it is in G, also acts on W (though it need not be in Y). We deal with the case when it does not act on W in the final section of this paper.

If $V|_X$ is irreducible, then we are in the case examined by Seitz in [10], with the additional condition that X have an outer automorphism which acts on V. We examine Table 1 of that paper, and find that we have such a situation in the examples there labelled I_4, I_5, I_6 for $n = 3$, II_1, S_1, S_8 (here we could take $G = X\langle t\rangle$, $G = X\langle s\rangle$, or $G = X\langle s,t\rangle$, where t,s are outer automorphisms of X of order 2 and 3 respectively), and MR_4. Henceforth we shall assume that $V|_X$ is not irreducible.

1991 *Mathematics Subject Classification.* 20G05.

Supported in part by the National Science Foundation.

The author would like to thank the referee for many detailed and helpful comments.

1.1. **Notation and Conventions.** All structures are assumed to be constructed over the same algebraically closed field K, of characteristic $p \geq 0$. Throughout, X will denote a simple algebraic group over K admitting an outer automorphism (so X is of type A_m, D_m, or E_6). A fixed standard graph automorphism of order 2 will be denoted by t, and if X has an outer automorphism of order 3 (i.e. if $X = D_4$), we will fix one and denote it by s. Thus G is $X\langle t\rangle$ except possibly when $X = D_4$, in which case we also consider $G = X\langle s\rangle$ and $G = X\langle s,t\rangle$.

We let B_X be a fixed t-stable Borel subgroup of X, containing a fixed t-stable maximal torus T_X. Define sets of simple roots $\{\beta_1, \beta_2, ..., \beta_m\} = \Pi(X)$ and fundamental dominant weights $\{\delta_1, \dots, \delta_m\}$ with respect to T_X and B_X, but with the opposite of the standard convention: $B_X = U_X T_X$ where $U_X = \prod U_{-\alpha}$ for $\alpha \in \Sigma^+(X)$. Then for $J \subseteq \Pi(X)$, P_X is the opposite of the standard parabolic corresponding to J. We assume the δ_i are numbered so that δ_i corresponds to β_i for every i. The set of roots of X is $\Sigma(X)$; the set of positive roots $\Sigma^+(X)$.

The group Y will be a simple algebraic group over K of classical type and rank n (A_n, B_n, C_n or D_n), such that $X < Y$ and $G \leq \mathrm{Aut}(Y)$. Let $\{\alpha_1, \alpha_2, ..., \alpha_n\} = \Pi(Y)$ be a set of simple roots of Y, and $\{\lambda_i\}$ the set of fundamental dominant weights such that λ_i corresponds to α_i. The set of roots of Y is $\Sigma(Y)$; the set of positive roots $\Sigma^+(Y)$. Notation and conventions similar to those used for X are used for parabolic subgroups of Y.

For a group H acting on a module M, $[M, H^l]$ will denote the l-fold commutator of H with M.

The K-vector space V is assumed to be a restricted irreducible Y-module with high weight $\lambda = \sum a_i \lambda_i$, such that V is irreducible as a G-module but not as an X-module (see the comment at the end of the previous subsection). We assume that the T_X-high weights of V are restricted as well. So if $G = X\langle t\rangle$, then $V|_X = V_1 \oplus V_2$, where each of V_1, V_2 is a restricted irreducible X-module.

The natural module for Y will be denoted by W. If W is irreducible as an X-module, then δ will denote its T_X-high weight. We will always assume that Y is the smallest of $\mathrm{SL}(W)$, $\mathrm{SO}(W)$, $\mathrm{Sp}(W)$ containing X. To justify this assumption, we must eliminate the situation when $X < Y < \mathrm{SL}(W), Y = \mathrm{SO}(W)$ or $\mathrm{Sp}(W)$, and V is a reducible Y-module, but an irreducible G-module. Assume we have such a situation.

If $X = D_4$ and $s \in G$, then $X\langle s\rangle < Y$ because Y has no outer automorphisms of order 3. In this case let $\tilde{X} = X\langle s\rangle$; otherwise set $\tilde{X} = X$. Now $V|_{\tilde{X}} = V_1 \oplus V_2$ and $V|_Y = V_1 \oplus V_2$ for some irreducible $\tilde{X}$- and Y-modules V_1 and V_2. Now if $\tilde{X} = X$, then since $X \neq Y$, the triple (V_1, X, Y) must appear in the list in [10]. Here, though, we have the extra hypotheses that $Y \neq A_\ell$, X has a graph automorphism not fixing V_1 (or V_2); both V_1 and V_2 appear in the table; and Y has an outer automorphism ($G \leq \mathrm{Aut}(Y)$ $t \notin Y$ since $V|_Y$ is not irreducible). There are no entries in [10] which satisfy all these conditions.

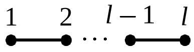
Similarly, if $\tilde{X} = D_4\langle s\rangle$, then (V_1, D_4, Y) would appear in Table 2, with $G = D_4\langle s\rangle$; there are no such examples there. So our standing assumption that Y is the smallest of $\mathrm{SL}(W), \mathrm{SO}(W)$, and $\mathrm{Sp}(W)$ containing X is justified.

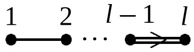
Except for the last section, we will assume that G acts on W (X clearly acts on W since $X \leq Y$, but t might not act on W). When $t \in G$ acts on W, $X\langle t\rangle$ fixes a nondegenerate bilinear form on W ([14, Lemma 79]); but in characteristic 2, it is conceivable that X fixes a quadratic form which t does not. Then we would have $X \leq \mathrm{SO}(W)$; $X\langle t\rangle \leq \mathrm{Sp}(W)$. Let q be the quadratic form defining $\mathrm{SO}(W)$; let q' be the quadratic form given by $q'(w) = q(w^t)$ for $w \in W$. Both forms are X-invariant.

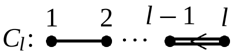
Lemma 1.1. *With the above setup, $q = q'$, so q is t-stable and $X\langle t\rangle \leq \mathrm{O}(W)$.*

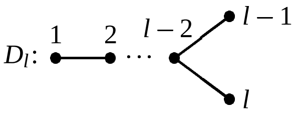
Proof. Let $R = \{w \in W | q(w) = q'(w)\}$. It is clear that R is a subspace of W and that R is X-invariant. But X acts irreducibly on W, so $R = 0$ or $R = W$. For a T_X-maximal vector $w^+ \in W$, $q(w^+) = 0 = q'(w^+)$ (let $h \in T_X$ such that $hw^+ = sw^+$ for some $s \in K^*$ such that $1 \neq s^2$ — then $q(w^+) = q(hw^+) = s^2 q(w^+)$, so $q(w^+) = 0$; similarly for q'). So $R = W$, which implies $q = q'$. ☐

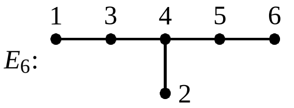
We label Dynkin diagrams for the groups we will be dealing with as follows, and we always number fundamental roots and fundamental dominant weights to agree with this labelling:

A_l:


B_l:








2. Q_X-LEVELS AND EMBEDDINGS OF PARABOLICS

In this section we introduce important facts about the "commutator series" of a module of a simple algebraic group.

Lemma 2.1. *If H is a simple algebraic group whose root system has only one root length, then restricted irreducible H-modules are tensor indecomposable (in particular, restricted irreducible X-modules are tensor indecomposable).*

Proof. This is part of 1.6 of [10]. ☐

Lemma 2.2. *Let M be an irreducible restricted H-module with high weight γ for some simple algebraic group H. Let P be a proper parabolic subgroup of H, with $P = QL$ a Levi decomposition. Then $M/[M,Q]$ is irreducible for L and for $L' = [L,L]$, with $T_{L'}$-high weight $\gamma|_{T_{L'}}$.*

Proof. This is 1.7 and 2.1 of [10]. ☐

Let H, M, γ, and P be as in the last lemma. Let $\{\varepsilon_i\}$ be the set of fundamental roots of H.

Definition 2.3. Let μ be a weight of M, say $\mu = \gamma - \sum c_i\varepsilon_i$, with each $c_i \geq 0$. The *Q-level* of μ is $\sum c_j$, where the sum ranges over those j for which $\varepsilon_j \in \Pi(H) - \Pi(L')$. The *$Q$-level l of M* is the sum of weight spaces for weights having Q-level l and is denoted M_l.

Lemma 2.4. *H, M, and P as above. If H is simply laced or if $p > 2$ (> 3 for $H = G_2$), then*

1. $[M,Q^l] = \oplus M_\mu$, *the sum taken over those weights μ having Q-level at least l.*
2. $[M,Q^l]/[M,Q^{l+1}] \cong M_l$
3. $\dim([M,Q^l]/[M,Q^{l+1}]) \leq s \cdot \dim([M,Q^{l-1}]/[M,Q^l])$, *where s is the number of positive roots β such that $U_{-\beta} \leq Q$ and $\beta = \varepsilon_i + \beta'$ for some $\varepsilon_i \in \Pi(H) - \Pi(L')$, with $\beta' = 0$ or a sum of roots in $\Pi(L')$.*
4. $\dim([M,Q^l]/[M,Q^{l+1}]) \leq \dim(Q) \cdot \dim([M,Q^{l-1}]/[M,Q^l])$.

Proof. This is 2.3 of [10]. ☐

We will write $M^l(Q_Y)$ for the quotient $[M,Q^{l-1}]/[M,Q^l]$.

Lemma 2.5. *Let $H = A_l$; let c be an integer such that $0 < c < p$; and let γ_1, γ_l be the "end" fundamental dominant weights for H. The irreducible module M having high weight $c\gamma_1$ or $c\gamma_l$ has all weight spaces of dimension 1; in particular,* $\dim(M) = (l+c)!/l!c!$.

Proof. This is 1.14 of [10]. ☐

We will occasionally use the Weyl character formula for dimensions of Weyl modules.

Finally, we claim that when X acts irreducibly on W, we may assume W is in fact restricted as an X-module.

Lemma 2.6. *If X acts irreducibly on W, then as an X-module, W has a restricted high weight.*

Proof. By Steinberg's tensor product theorem ([13]), $W =_X W_1^{q_1} \otimes W_2^{q_2} \otimes \cdots \otimes W_r^{q_r}$ where the W_i are restricted irreducible X-modules and $q_1, \ldots, q_r$ are distinct powers of p. Let X_i be the isometry group of $W_i^{q_i}$. Then the embedding of X in Y, which is given by the action on W, factors through an embedding $X \hookrightarrow X_1 X_2 \ldots X_r \leq Y$ given by $x \mapsto (x^{q_1}, x^{q_2}, \ldots, x^{q_r})$. But then the action of X on V factors through this same embedding (since the action of X on V is given by the embedding of X in Y). Notice that each X_i must act nontrivially on V since Y is simple. Now if $V|_{X_1 X_2 \ldots X_r}$ is irreducible, then as an X-module, V has a high weight given by the restriction of the $X_1 X_2 \ldots X_r$-high weight to T_X; but this gives a T_X-high weight of the form $q_1 \gamma_1 + \cdots + q_r \gamma_r$ for some non-zero restricted T_X-weights γ_i. This is impossible unless $r = 1$ and $q_1 = 1$, as the T_X-high weights of V are restricted. Similarly, if V is reducible as an $X_1 X_2 \ldots X_r$-module and there is an i such that $q_i \neq 1$, then let V_i be a $X_1 X_2 \ldots X_r$-subquotient of V on which X_i acts nontrivially. Then as an X-module, V_i has a non-restricted high weight as above, again a contradiction. $\square$

Now let P_X be a parabolic subgroup of X, and $P_X = Q_X L_X$ a Levi decomposition with $T_X \leq L_X$ (if P_x is t-stable, choose L_X to also be t-stable). Assume that X acts irreducibly on W with high weight δ, which is restricted by the Lemma above. We wish to give a construction of a parabolic subgroup P_Y of Y (with $P_Y = Q_Y L_Y$ a Levi decomposition) such that $P_X \leq P_Y$, $Q_X \leq Q_Y$, $L_X \leq L_Y$. Let $Z = Z(L_X)^\circ$.

Lemma 2.7. *The stabilizer in Y of the commutator series*

$$W > [W, Q_X] > [W, Q_X, Q_X] > \cdots > 0$$

is a parabolic subgroup P_Y of Y satisfying the following:

1. *$P_X \leq P_Y$ and $Q_X \leq Q_Y = R_u(P_Y)$.*
2. *$L_Y = C_Y(Z)$ is a Levi factor of P_Y containing L_X.*
3. *If T_Y is a maximal torus of Y containing T_X, then $T_Y \leq L_Y$.*

Proof. Since X acts irreducibly on W, we can apply the Q_X-level construction above to W and P_X. Suppose $Y = \mathrm{Sp}(W)$ or $\mathrm{SO}(W)$. Then $W \cong W^*$ and we identify these modules via the inner product. So the fixed point sets $W(Q_X)$ and $W^*(Q_X)$ are equal, and a trivial calculation shows that $W^*(Q_X)$ is the annihilator in W^* of $[W, Q_X]$. But from 1.2 in [10] it follows that $W(Q_X) = [W, Q_X^k]$, where k is minimal with respect to $[W, Q_X^{k+1}] = 0$. By induction, $[W, Q_X^i]$ annihilates $[W, Q_X^{k-i+1}]$.

By Lemma 2.4, $[W, Q_X^l]/[W, Q_X^{l+1}] \cong \sum_\mu W_\mu = W_l$, with the sum taken over those weights μ of Q_X-level l, so by the last paragraph we have $(W_i, W_j) = 0$ unless $i + j \leq k$. Let $c = [(k-1)/2]$. Then $0 < W_k < W_k \oplus W_{k-1} < \cdots < W_k \oplus \cdots \oplus W_{k-c}$ is a flag of totally isotropic subspaces of W, so its stabilizer P_Y in Y is a parabolic subgroup. But our discussion of weights shows this flag is just the flag $0 < [W, Q_X^k] < [W, Q_X^{k-1}] < \cdots < [W, Q_X^{k-c}]$, and if we adjoin annihilators, we obtain the full commutator series $0 < [W, Q_X^k] < \cdots < [W, Q_X] < W$. Hence P_Y is the stabilizer of this commutator series. Let $Q_Y = R_u(P_Y)$.

Let $Z = Z(L_X)^\circ$. Let $\tilde{L}$ be a Levi factor of P_Y containing Z, and let $L_Y = C_Y(Z) > L_X$. Then Z induces scalars on $W/[W, Q_X]$ since this module is an irreducible L_X-module, and so $[Z, \tilde{L}] \leq \tilde{L}$ has a trivial action on $W/[W, Q_X]$, which implies $[Z, \tilde{L}] \leq Q_Y$. But $Q_Y \cap \tilde{L} = 1$, so $\tilde{L} \leq C_Y(Z) = L_Y$. This implies $\tilde{L} = L_Y$ as $\tilde{L}$ is a maximal reductive subgroup of P_Y. Let T_Y be a maximal torus of Y containing T_X. Then $T_Y \leq C_Y(T_X) \leq C_Y(Z) = L_Y$.

Note that $[W, Q_Y^l] = [W, Q_X^l]$ for every l by construction. If $u \in U_{-\alpha}$ for $\alpha \in \Pi(X) - \Pi(L_X)$, then $uW_{\delta - e_1\beta_1 - e_2\beta_2 - \ldots} \subseteq \sum_{l \geq 0} W_{\delta - e_1\beta_1 - e_2\beta_2 - \cdots - l\alpha}$. So $Q_X \leq Q_Y$. Since P_X stabilizes each factor in the flag, $P_X \leq P_Y$.

If $Y = SL(W)$, the argument is easier: The flag $0 < [W, Q_X^k] < [W, Q_X^{k-1}] < \cdots < [W, Q_X] < W$ determines a parabolic subgroup P_Y and the above arguments hold. $\square$

We give more information about this embedding for particular groups X and parabolic subgroups P_X below and in subsequent sections. For the next two Lemmas, we assume that $t \in G$ (where t is the fixed outer automorphism of X) and $V =_X V_1 \oplus V_2$, with V_1, V_2 irreducible X-modules.

Lemma 2.8. *If P_X is a t-stable parabolic subgroup of X and P_X is embedded in a parabolic subgroup P_Y of Y as above, then P_Y is likewise t-stable.*

Proof. This is clear if t acts on W, for then each subspace W_l of W is t-stable (since, if X is of type A_m for example, $W^t_{\delta-e_1\beta_1-e_2\beta_2-\cdots-e_m\beta_m} = W_{\delta-e_m\beta_1-e_{m-1}\beta_2-\cdots-e_1\beta_m}$). So P_Y^t is the stabilizer in W of the same flag as is P_Y; i.e. $P_Y = P_Y^t$.

Now assume t does not act on W. Then, since t acts on Y, we have $Y = A_n$ (as outer automorphisms of D_n preserve the natural module). Let the Q_X-level of the T_X-low weight of W be k as above. The dimensions of Q_X-levels of W are symmetric about $k/2$; that is, $\dim(W_i) = \dim(W_{k-i})$, since they are interchanged by a representative in X of the long word of the Weyl group. This means that P_Y is symmetric; i.e. $\Pi(L_Y')$ is preserved under the automorphism of the Dynkin diagram of Y. So there is a graph automorphism σ of Y which preserves P_Y; since all graph automorphisms may be written as the product of t with an element of Y, there is a $g \in Y$ such that $t = \sigma g$. Then $P_Y^t = P_Y^{\sigma g} = P_Y^g$. So P_Y and P_Y^t are conjugate.

Since P_Y is the stabilizer of the flag $W > [W, Q_Y] > \dots$, $P_Y^t = P_Y^g$ is the stabilizer of $W > [W, Q_Y^g] > [W, Q_Y^g, Q_Y^g] > \dots$. Also, $P_X = P_X^t \le P_Y^t$, so $Q_X = Q_X^t \le Q_Y^t = Q_Y^g$. This gives $[W, Q_Y^i] = [W, Q_X^i] \le [W, (Q_Y^g)^i]$ for every i. But $\dim([W, Q_Y^i]) = \dim([W, (Q_Y^g)^i])$; so in fact $[W, (Q_Y^g)^i] = [W, Q_Y^i]$ for every i. But this implies that P_Y and P_Y^t are the stabilizers of the same flag in W, or $P_Y = P_Y^t$. ☐

Lemma 2.9. *If P_X is a t-stable parabolic subgroup of X and P_X is embedded in a parabolic subgroup P_Y of Y as above, then $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$.*

Proof. The involution t interchanges the two T_X-high weight 1-spaces $\langle v_1\rangle \subseteq V_1$ and $\langle v_2\rangle \subseteq V_2$ of V. We have $[V, Q_X] \le [V, Q_Y]$ since $Q_X \le Q_Y$, and $V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$, with each summand an irreducible L_X'-module by Lemma 2.2.

Since L_X is contained in L_Y and $V/[V, Q_Y]$ is an irreducible L_Y-module, either $(V/[V, Q_Y])|_{L_X} \cong V_i/[V_i, Q_X]$ for $i = 1$ or 2, or $(V/[V, Q_Y])|_{L_X} = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$. If the former, say $(V/[V, Q_Y])|_{L_X} \cong V_1/[V_1, Q_X]$, then $v_2 \in [V, Q_Y]$ (notice that $V_1/[V_1, Q_X] \not\cong V_2/[V_2, Q_X]$ as L_X-modules because $T_X \le L_X$ acts differently on a high weight vector $v_1 \in V_1$ than on a high weight vector $v_2 \in V_2$, and v_i has a non-zero image in $V_i/[V_i, Q_X]$). But Q_Y is t-stable by the last lemma (Q_Y is a characteristic subgroup of P_Y), so $[V, Q_Y]$ is t-stable. This would imply $\langle v_2^t\rangle = \langle v_1\rangle \subseteq [V, Q_Y]$, which is impossible. So $(V/[V, Q_Y])|_{L_X} = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$. ☐

Let $P_Y = L_Y Q_Y$ be a parabolic subgroup of Y. For each $\gamma \in \Pi(Y) - \Pi(L_Y')$, we define a certain normal subgroup K_Y^γ of P_Y, as in [10, page 44]: Let $\Sigma_\gamma(Y)$ denote the set of roots in $\Sigma(Y)$ having γ-coefficient -1 and zero coefficient for other roots in $\Pi(Y) - \Pi(L_Y')$. Then let K_Y^γ be the product of those T_Y-root subgroups U_β for $\beta \in \Sigma^-(Y) - \Sigma^-(L_Y') - \Sigma_\gamma(Y)$. From the commutator relations it follows that K_Y^γ is normal in P_Y and we let $Q_Y^\gamma = Q_Y/K_Y^\gamma$. This construction also applies to a parabolic subgroup P_X of X. In particular, if P_X is a maximal parabolic subgroup corresponding to $\alpha \in \Pi(X)$, then set $Q_X^\alpha = Q_X/K^\alpha$, where K^α is the product of those T_X-root subgroups corresponding to roots having α-coefficient strictly less than -1.

Lemma 2.10. *If $P_X = Q_X L_X$ is a maximal parabolic subgroup corresponding to $\alpha \in \Pi(X)$, then:*

1. $K^\alpha = [Q_X, Q_X]$.
2. *Q_X^α is an irreducible L_X'-module with $-\alpha$ as its $T_{L_X'}$-high weight.*

Proof. See 3.2 in [10] (remembering that X is of type A_m, D_m, or E_6). ☐

Again assume $t \in G$. Let P_X be a parabolic subgroup of X (not necessarily t-stable) containing the fixed t-stable Borel subgroup B_X. Embed P_X in a parabolic subgroup P_Y of Y via the above construction. Write $L_Y' = L_1 \times \cdots \times L_r$, a direct product of simple groups. By Lemma 2.2, L_Y' acts irreducibly on $V^1(Q_Y) = V/[V, Q_Y]$. Then $V^1(Q_Y) = V^1 \otimes \cdots \otimes V^r$ where for each i, V^i is an irreducible module for L_i. The embedding $L_X \to L_Y$ gives an embedding of L_X' into $L_1 \times \cdots \times L_r$, and via the projections $L_X' \to L_i$, any L_i-module, in particular V^i, can be regarded as a module for L_X'.

Since $Q_X \le Q_Y$, we have $[V, Q_X] \le [V, Q_Y]$ and hence $V/[V, Q_Y]$ is a quotient of $V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$, with each of these summands irreducible L_X'-modules. Since $L_X' \le L_Y'$, this implies that either $V/[V, Q_Y]$ is irreducible for L_X', or $V/[V, Q_Y] = V/[V, Q_X] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$. Lemma 2.9 tells us that the latter happens when P_X is t-stable.

Lemma 2.11. *If V, $P_X = L_X Q_X$, $P_Y = L_Y Q_Y$, and L_i are as above with P_X t-stable, then only one L_i acts nontrivially on $V/[V,Q_Y]$.*

Proof. By Lemma 2.9, $V/[V,Q_Y] = V/[V,Q_X] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X]$. Let V^i be the obvious module for L_i as above (i.e. if L_i corresponds to $\{\alpha_j,\alpha_{j+1},...,\alpha_k\} \subseteq \Pi(Y)$, then V^i is the L_i-module with high weight $a_j\lambda_j + a_{j+1}\lambda_{j+1} + \cdots + a_k\lambda_k$). Then $\otimes_i V^i|_{L'_X} = V/[V,Q_Y] = V_1/[V_1,Q_X] \oplus V_2/[V_2,Q_X] = V'_1 \oplus V'_2$, with the latter two (restricted) irreducible L'_X-modules. Notice that $V/[V,Q_Y]$ has no tensor decomposable L'_X-submodules, since the only L'_X-submodules of $V/[V,Q_Y]$ are isomorphic to V'_1 or V'_2, which are both irreducible L'_X-modules, and Lemma 2.1 implies that no irreducible L'_X-module is tensor decomposable.

None of the V^i can be reducible as L'_X-modules: Assume V^1 has an L'_X-submodule V'. Then $V' \otimes V^2 \otimes \dots$ is a proper tensor decomposable L'_X-invariant subspace of $V/[V,Q_Y]$, which is impossible by the above. So each of the V^i is an irreducible L'_X-module; they are restricted as well, since the sum of their L'_X-high weights must be an L'_X-high weight in $V/[V,Q_Y]$, and these are both restricted by assumption.

Similarly, assume there are more than two V^i. Then $V^1 \otimes V^2$ is not an irreducible L'_X-module by Steinberg's tensor product theorem ([13]) (since there is no twist in the embedding $L'_X \hookrightarrow L_i$), so it must have a proper L'_X-invariant subspace V'. But then $V' \otimes V^3 \otimes \dots$ is a proper tensor decomposable L'_X-invariant subspace of $V/[V,Q_Y]$, which is again a contradiction.

So assume two V^i are nontrivial; we have $V^1 \otimes V^2 = V'_1 \oplus V'_2$, all restricted irreducible L'_X-modules. Let λ^i be the $T_Y \cap L_i = T_{L_i}$-high weight of V^i; γ_i the $T_X \cap L'_X = T_{L'_X}$-high weight of V'_i. The Levi factor $L_Y = C_Y(Z(L_X)^\circ)$ is t-stable since L_X is; hence L'_Y is t-stable. Note that each L_i has rank greater than 1 (since L'_X projects nontrivially to L_i). Thus the subsystem groups L_1 and L_2 are either interchanged by t (if Y has type A_n and $t \notin Y$) or fixed by t (if $t \in Y$ or Y has type other than A_n). Let v^1 (respectively v^2) be a $T_{L'_X}$-high weight vector of V^1 (respectively V^2), with respect to $B_X \cap L'_X = B_{L'_X}$ (a Borel subgroup of L'_X). Then $v^1 \otimes v^2$ is a $T_{L'_X}$-high weight vector of $V^1 \otimes V^2 = V/[V,Q_Y]$, and since $Z = Z(L_X)^\circ$ induces scalars on $V/[V,Q_Y]$, $v^1 \otimes v^2$ is in fact a $T_{L_X} = T_X$-high weight vector in $V/[V,Q_Y]$.

Assume that $Y = A_n$ and $t \notin Y$, so $L_1^t = L_2$. Then λ is symmetric with respect to the graph automorphism (since t acts on V), so $(v^1)^t \in Kv^2$ and $(v^2)^t \in Kv^1$ (as t stabilizes $B_{L'_X}$ and $T_{L'_X}$).

If L_1 and L_2 are fixed by t, then t preserves each of V_1 and V_2; hence, as t stabilizes $B_{L'_X}$ and $T_{L'_X}$, it stabilizes Kv^1 and Kv^2. So in any case, $(v^1 \otimes v^2)^t \in K(v^1 \otimes v^2)$, which is a contradiction as the two T_X-high weight 1-spaces in $V^1 \otimes V^2$ are interchanged by t. So one of the V^i must be trivial. □

One final crucial lemma, due to Suprunenko:

Lemma 2.12. *Let H be a simple algebraic group of type A_l, and M an irreducible H-module with restricted high weight γ. Let $\mathcal{W}$ be the Weyl module for H with high weight γ. Assume that μ is a weight such that $\mathcal{W}_\mu \neq 0$. Then $M_\mu \neq 0$.*

Proof. This is the result of [15]. □

3. THE $D_n < B_n$ CASE

In Section 4, we will present the proof of Theorem 1 for the case when $W|_X$ is reducible. It turns out that the hardest case will be when $X = D_n < B_n = Y$, so we do that case first. So we assume $X = D_n$, $G = X\langle t\rangle$, $Y = B_n$, with X contained in Y in the usual way.

Notation is as before. For H an algebraic group, let $L(H)$ denote the Lie algebra of H. For a simple Lie algebra with root system Φ having basis $\{\alpha_1,...,\alpha_m\}$, we use the standard Chevalley basis $\{e_\alpha, f_\alpha, h_i | \alpha \in \Phi^+, 1 \leq i \leq m\}$, satisfying the usual relations (in particular, $[e_{\alpha_i} f_{\alpha_i}] = h_i$). The usual ordering on weights will be denoted by $\succ$.

Throughout this section, X is a simple algebraic group over K of type D_n, embedded in $Y = B_n$ in the usual way (as the derived group of the stabilizer of a 1-space).

First we prove a proposition for irreducible representations of simple Lie algebras. The methods used in the proof of the proposition have an interesting application to the representation theory of the symmetric groups; see [6].

3.1. **Lie Algebra Representations.** Let V be an irreducible $L(B_n)$-module, with high weight $\lambda = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n$, $a_n \neq 0$ (λ_i the fundamental dominant weight corresponding to the root α_i), and high weight vector v^+. Assume $p \neq 2$. For the weight $\mu = \lambda - (\alpha_i + \cdots + \alpha_k)$ $(k \geq i)$, V_μ is spanned by vectors of the form

$$f_{\alpha_i+\cdots+\alpha_{i+l}} f_{\alpha_{i+l+1}+\cdots+\alpha_{i+l'}} \cdots f_{\alpha_{i+l^{(m)}+1}+\cdots+\alpha_k} v^+, \tag{$\dagger$}$$

where every f_β involves some α_j such that $a_j \neq 0$. For such a $k \geq i$, let $V_{i,k}$ denote the span of all of the above terms except for $f_{\alpha_i+\cdots+\alpha_k} v^+$; so $V_{i,k} \subseteq V_\mu$.

Proposition 3.1. *Let V and λ be as above. Let a_i, a_m be non-zero labels with $m > i$. If $f_{\alpha_r+\cdots+\alpha_m} v^+ \in V_{r,m}$ for all $i \leq r < m$, then $f_{\alpha_i+\cdots+\alpha_j} v^+ \in V_{i,j}$, where a_j is the first non-zero label after a_i.*

Proof. Assume the hypotheses. If $j = m$, the Proposition is vacuous; so assume there are non-zero coefficients between a_i and a_m; we proceed by induction on the number of such non-zero coefficients. Let a_k be the last non-zero coefficient before a_m.

For the calculations below, recall that since v^+ is a maximal vector, $e_\beta v^+ = 0$ for any $\beta \in \Sigma^+$, and that e_β and f_δ commute whenever $\beta - \delta \notin \Sigma$. If $\beta - \delta \in \Sigma$, then $e_\beta f_\delta = f_\delta e_\beta + d e_{\beta-\delta}$ for some $d = N(\beta, -\delta) \in \mathbb{Z}$ (where for $\alpha \in \Sigma^-$, $e_\alpha = f_{-\alpha}$).

Lemma 3.2. $f_{\alpha_r+\cdots+\alpha_k} v^+ \in V_{r,k}$ *for all* $i \leq r < k$.

Proof. Assume that there is an $r \in [i, k-1]$ such that $f_{\alpha_r+\cdots+\alpha_k} v^+ \notin V_{r,k}$.

To say $f_{\alpha_r+\cdots+\alpha_m} v^+ \in V_{r,m}$ is to say that there is a non-zero linear combination

$$0 = f_{\alpha_r+\cdots+\alpha_m} v^+ + \sum_{(\rho_1,\ldots,\rho_s)} d_{(\rho_1,\ldots,\rho_s)} f_{\rho_1} \cdots f_{\rho_s} v^+, \tag{$*$}$$

where $f_{\rho_1} \ldots f_{\rho_s} v^+$ is a term of type ($\dagger$), with $s \geq 2$ and $\sum \rho_l = \alpha_r + \cdots + \alpha_m$.

Now consider those summands of the right-hand side of ($*$) which give a multiple of $f_{\alpha_r+\cdots+\alpha_k} v^+$ when the product $e_{\gamma_1} \ldots e_{\gamma_q}$ is applied, where $\gamma_1 = \alpha_{k+1} + \cdots + \alpha_{k+l_1}, \gamma_2 = \alpha_{k+l_1+1} + \ldots$ and $\sum \gamma_l = \alpha_{k+1} + \cdots + \alpha_m$. Since each f_ρ must involve some α_l with $a_l \neq 0$, these terms are exactly the

$$F_l v^+ = f_{\alpha_r+\cdots+\alpha_{k+l}} f_{\alpha_{k+l+1}+\cdots+\alpha_m} v^+, \quad 0 \leq l \leq m-k,$$

since all the coefficients between a_k and a_m are 0. Notice that such a product of e_γ's applied to a summand of ($*$) gives a multiple of a generator of type ($\dagger$).

Let the coefficient of $F_l v^+$ in ($*$) be b_l $(= d_{(\alpha_r+\cdots+\alpha_{k+l}, \alpha_{k+l+1}+\cdots+\alpha_m)})$; note that $b_{m-k} = 1$. Since by our assumption $f_{\alpha_r+\cdots+\alpha_k} v^+$ is not a combination of other terms ($\dagger$) which appear in the $\lambda - (\alpha_r + \cdots + \alpha_k)$ weight space, any $e_{\gamma_1} \ldots e_{\gamma_q}$ (γ_l as in the last paragraph) must kill $\sum b_l F_l v^+$. In particular,

$$E_s = e_{\alpha_{k+1}+\cdots+\alpha_{k+s}} e_{\alpha_{k+s+1}+\cdots+\alpha_m}, \quad 0 \leq s < m-k,$$

must kill this sum.

Now assume $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \notin V_{r,k-1}$. Consider the summands in ($*$) which give a multiple of $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+$ when some $e_{\gamma_1} \ldots e_{\gamma_q}$ (where $\gamma_1 = \alpha_k + \cdots + \alpha_{k+l_1}, \gamma_2 = \alpha_{k+l_1+1} + \ldots$, and $\sum \gamma_l = \alpha_k + \cdots + \alpha_m$) is applied. They are the $F_l v^+$ $(0 \leq l \leq m-k)$, and

$$G_l v^+ = f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k+\cdots+\alpha_{k+l}} f_{\alpha_{k+l+1}+\cdots+\alpha_m} v^+, \quad 0 \leq l < m-k$$

(note that we need not include $G_{m-k} = f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k+\cdots+\alpha_m} v^+$ here by our assumption that $f_{\alpha_k+\cdots+\alpha_m} v^+ \in V_{k,m}$).

By the assumption that $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \notin V_{r,k-1}$, it follows that $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \neq 0$. So any $e_{\gamma_1} \ldots e_{\gamma_q}$ (γ_l as in the last paragraph) must kill

$$\sum_{l=0}^{m-k} b_l F_l v^+ + \sum_{l=0}^{m-k-1} c_l G_l v^+,$$

where c_l is the coefficient of G_l in $(*)$. In particular, $e_{\alpha_k}E_s$ must kill $\sum c_l G_l v^+$ for every $0 \leq s < m-k$. The following shows that $E_s G_l v^+ = 0$ if $s \neq l$: Assume $s < l$. Then

$$\begin{aligned}
E_s G_l v^+ &= (e_{\alpha_{k+1}+\cdots+\alpha_{k+s}} e_{\alpha_{k+s+1}+\cdots+\alpha_m} f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k+\cdots+\alpha_{k+l}} f_{\alpha_{k+l+1}+\cdots+\alpha_m}) v^+ \\
&= e_{\alpha_{k+1}+\cdots+\alpha_{k+s}} (f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k+\cdots+\alpha_{k+l}} e_{\alpha_{k+s+1}+\cdots+\alpha_m} f_{\alpha_{k+l+1}+\cdots+\alpha_m} v^+) \\
&= d e_{\alpha_{k+1}+\cdots+\alpha_{k+s}} (f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k+\cdots+\alpha_{k+l}} e_{\alpha_{k+s+1}+\cdots+\alpha_{k+l}} v^+) = 0
\end{aligned}$$

for some structure constant d. A similar calculation holds for $s > l$. Also,

$$\begin{aligned}
e_{\alpha_k} E_l G_l v^+ &= e_{\alpha_k} f_{\alpha_r+\cdots+\alpha_{k-1}} e_{\alpha_{k+1}+\cdots+\alpha_{k+l}} f_{\alpha_k+\cdots+\alpha_{k+l}} e_{\alpha_{k+l+1}+\cdots+\alpha_m} f_{\alpha_{k+l+1}+\cdots+\alpha_m} v^+ \\
&= \pm e_{\alpha_k} f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k} h_{\alpha_{k+l+1}+\cdots+\alpha_m} v^+ \\
&= \pm a_k a_m f_{\alpha_r+\cdots+\alpha_{k-1}} v^+.
\end{aligned}$$

So:

$$\begin{aligned}
0 = e_{\alpha_k} E_0 (\textstyle\sum c_l G_l v^+) &= \pm c_0 a_k a_m f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \Rightarrow c_0 = 0 \\
0 = e_{\alpha_k} E_1 (\textstyle\sum c_l G_l v^+) &= \pm c_1 a_k a_m f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \Rightarrow c_1 = 0 \\
&\vdots \\
0 = e_{\alpha_k} E_{m-k-1} (\textstyle\sum c_l G_l v^+) &= \pm c_{m-k-1} a_k a_m f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \Rightarrow c_{m-k-1} = 0.
\end{aligned}$$

So this implies $\sum c_l G_l v^+ = 0$. But then:

$$\begin{aligned}
0 &= e_{\alpha_k+\cdots+\alpha_m} (\textstyle\sum b_l F_l v^+ + \sum c_l G_l v^+) = e_{\alpha_k+\cdots+\alpha_m} (\sum b_l F_l v^+) \\
&= d f_{\alpha_r+\cdots+\alpha_{k-1}} v^+
\end{aligned}$$

with

$$d = \begin{cases} \pm 2 & \text{if } m = n \\ \pm 1 & \text{otherwise} \end{cases}$$

(as $F_{m-k} v^+ = f_{\alpha_r+\cdots+\alpha_m} v^+$ is the only term in $\sum b_l F_l v^+$ not killed by $e_{\alpha_k+\cdots+\alpha_m}$). But this is a contradiction, since we have assumed $p \neq 2$.

So our assumption that there was an r, $i \leq r < k$, such that $f_{\alpha_r+\cdots+\alpha_k} v^+ \notin V_{r,k}$ and $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \notin V_{r,k-1}$ must be false; i.e. for every r, $i \leq r < k$, either (i) $f_{\alpha_r+\cdots+\alpha_k} v^+ \in V_{r,k}$, or (ii) $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \in V_{r,k-1}$. We want to show that in fact (i) holds always. There are several cases:

(a) If $r < k-1$ and $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ \in V_{r,k-1}$, we see that:

$$\begin{aligned}
f_{\alpha_r+\cdots+\alpha_k} v^+ &= f_{\alpha_k} f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ - f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k} v^+ \\
&= \sum_{(\rho_1,\ldots,\rho_m), m \geq 2} f_{\alpha_k} f_{\rho_1} \cdots f_{\rho_m} v^+ - f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k} v^+ \\
&= \sum f_{\rho_1} \cdots f_{\rho_{m-1}} f_{\alpha_k} f_{\rho_m} v^+ - f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k} v^+ \\
&= \sum f_{\rho_1} \cdots f_{\rho_m} f_{\alpha_k} v^+ + s f_{\rho_1} \cdots f_{\rho_m+\alpha_k} v^+ - f_{\alpha_r+\cdots+\alpha_{k-1}} f_{\alpha_k} v^+
\end{aligned}$$

for some integers s (depending on $(\rho_1, \ldots, \rho_m)$), so $f_{\alpha_r+\cdots+\alpha_k} v^+ \in V_{r,k}$.

(b) If $r = k-1$ and $a_{k-1} \neq 0$, (ii) cannot happen, since $V_{k-1,k-1} = 0$ and $f_{\alpha_r+\cdots+\alpha_{k-1}} v^+ = f_{\alpha_{k-1}} v^+ \neq 0$.

(c) If $r = k-1$ and $a_{k-1} = 0$, then

$$f_{\alpha_{k-1}+\alpha_k} v^+ = \pm (f_{\alpha_k} f_{\alpha_{k-1}} v^+ - f_{\alpha_{k-1}} f_{\alpha_k} v^+) = \pm f_{\alpha_{k-1}} f_{\alpha_k} v^+,$$

since $a_{k-1} = 0$. So $f_{\alpha_r+\cdots+\alpha_k} v^+ = f_{\alpha_{k-1}+\alpha_k} v^+ \in V_{k-1,k} = V_{r,k}$.

So the Lemma is proved. □

Since there are fewer non-zero labels between a_i and a_k than between a_i and a_m, by our inductive hypothesis $f_{\alpha_i+\cdots+\alpha_j} v^+ \in V_{i,j}$. So the Proposition is proved. □

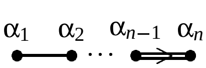
3.2. **The case $D_n < B_n$.** Assume $X = D_n$ is embedded in $Y = B_n$ in the usual way (as the derived group of the stabilizer of a 1-space in the natural module for Y). Now V is a restricted irreducible KB_n-module, with high weight $\lambda = a_1\lambda_1 + a_2\lambda_2 + \cdots + a_n\lambda_n$ (recall that λ_i is the fundamental dominant weight corresponding to the root α_i of $Y = B_n$) and high weight vector v^+. Then D_n is the subgroup of B_n generated by all the root subgroups corresponding to long roots, and t may be chosen to be a representative in B_n of the Weyl group reflection s_{α_n}. The K-vector space V is irreducible as a $KD_n\langle t\rangle$- module, but not as a KD_n-module. The high weights of V as a D_n-module are restricted. The symbol "$\equiv$" will mean congruent modulo p.

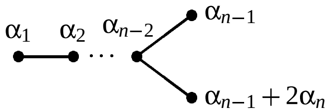
Theorem 3.3. *If $p \neq 2$, then V restricted to $KD_n\langle t\rangle$ is irreducible if and only if $a_n = 1$, $\langle \lambda+\rho, \alpha_i + \cdots + \alpha_j\rangle \equiv 1$ for every pair of successive non-zero coefficients a_i, a_j with $i < j \leq n$. If $p = 2$, then V restricted to $KD_n\langle t\rangle$ is irreducible if and only if $a_n = 1$ and $a_i = 0$ for $i < n$.*

Note that in any characteristic, the spin module (high weight λ_n) for B_n is the sum of two spin modules for D_n.

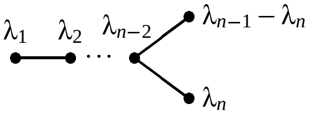
Proof. We will work over the Lie algebras of X and Y, rather than the group algebras. By Lemma 1.1 ii) of [10], V is an irreducible $L(B_n)$-module. The notation for elements of the Lie algebra was introduced in the previous section. The proof will consist of a series of Lemmas.

The following are the roots which correspond to nodes of the Dynkin diagrams for X and Y (since we know the embedding $X \hookrightarrow Y$ precisely, and X contains the full maximal torus of Y, we know what the β_i are in terms of the α_j):





The fundamental dominant weights for X are



where $\lambda_1, \lambda_2, ..., \lambda_n$ are the fundamental dominant weights for B_n.

Lemma 3.4. *Assume that $a_n \leq 1$ and $p \neq 2$. Then $V = L(D_n)v^+ \oplus L(D_n)f_{\alpha_n}v^+$ as a KD_n-module if and only if $f_{\alpha_i+\cdots+\alpha_n}v^+ \in L(D_n)f_{\alpha_n}v^+$ and $f_{\alpha_i+\cdots+\alpha_n}f_{\alpha_n}v^+ \in L(D_n)v^+$ for all $i < n$.*

Proof. Note that $f_{\alpha_n}v^+$ is a maximal vector for D_n, and as mentioned above D_n contains the full maximal torus of B_n.

$\Rightarrow$: First: $f_{\alpha_i+\cdots+\alpha_n}v^+ \in L(D_n)f_{\alpha_n}v^+$. This is clear if $f_{\alpha_i+\cdots+\alpha_n}v^+ = 0$ for every i. If this is not the case, let $i < n$ such that $0 \neq w = f_{\alpha_i+\cdots+\alpha_n}v^+$. Then w is a weight vector of weight $\mu = \lambda - (\alpha_i + \cdots + \alpha_n)$. So if $w = w_1 + w_2$, with $w_1 \in L(D_n)v^+$ and $w_2 \in L(D_n)f_{\alpha_n}v^+$, then w_1, w_2 are both also weight vectors of weight $\lambda - (\alpha_i + \cdots + \alpha_n)$; so $w_1 = 0$, since all weights appearing in $L(D_n)v^+$ are of the form $\lambda - \beta$ for $\beta \in \langle \alpha_1, ..., \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\rangle$. Thus $w = w_2 \in L(D_n)f_{\alpha_n}v^+$. Similar arguments show that $f_{\alpha_i+\cdots+\alpha_n}f_{\alpha_n}v^+ \in L(D_n)v^+$.

$\Leftarrow$: Let $V_1 = L(D_n)v^+$ and $V_2 = L(D_n)f_{\alpha_n}v^+$. We claim that if $f_{\alpha_i+\cdots+\alpha_n}v^+ \in V_1 \oplus V_2$ and $f_{\alpha_i+\cdots+\alpha_n}f_{\alpha_n}v^+ \in V_1 \oplus V_2$ for all $i < n$, then $V = V_1 \oplus V_2$.

Proof (claim). a) First we prove that the hypotheses of the claim imply

$$f_{\gamma_1} \cdots f_{\gamma_s} f_{\alpha_n}^a v^+ \in V_1 \oplus V_2,$$

whenever $a = 0$ or 1, γ_1 is a short root, and $\gamma_2, ..., \gamma_s$ are long. Assume this is not the case, and let $f_{\gamma_1} \cdots f_{\gamma_m} f_{\alpha_n}^a v^+$ be a counterexample with m minimal. By the hypothesis, $m \geq 2$. Then

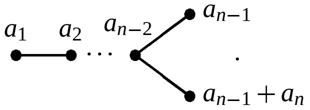
$$f_{\gamma_1} \cdots f_{\gamma_m} f_{\alpha_n}^a v^+ = f_{\gamma_2} f_{\gamma_1} f_{\gamma_3} \cdots f_{\gamma_m} f_{\alpha_n}^a v^+ + N(\gamma_1, \gamma_2) f_{\gamma_1+\gamma_2} f_{\gamma_3} \cdots f_{\gamma_m} f_{\alpha_n}^a v^+,$$

and $f_{\gamma_1} f_{\gamma_3} \dots f_{\gamma_m} f^a_{\alpha_n} v^+$ (thus $f_{\gamma_2} f_{\gamma_1} f_{\gamma_3} \dots f_{\gamma_m} f^a_{\alpha_n} v^+$) and $f_{\gamma_1+\gamma_2} f_{\gamma_3} \dots f_{\gamma_m} f^a_{\alpha_n} v^+$ are in $V_1 \oplus V_2$ by the minimality of m. But this contradicts our assumption that $f_{\gamma_1} \dots f_{\gamma_m} f^a_{\alpha_n} v^+ \notin V_1 \oplus V_2$. So a) is proved.

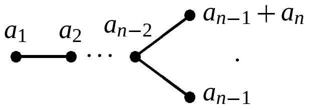
b) The Y-module V is spanned by elements of the form $f_{\beta_1} \dots f_{\beta_\ell} v^+$, with the β_i roots of $Y = B_n$. Assume the claim is false, and let $f_{\beta_1} \dots f_{\beta_m} v^+$ be one of these spanning elements not in $V_1 \oplus V_2$, with m minimal. By minimality, $f_{\beta_1} \dots f_{\beta_m} v^+ = f_{\beta_1}(\sum a f_{\gamma_1} \dots f_{\gamma_s} v^+ + \sum b f_{\varepsilon_1} \dots f_{\varepsilon_t} f_{\alpha_n} v^+)$, where all the γ_j, ε_j are long roots. By a), and our assumption (that $f_{\beta_1} v^+, f_{\beta_1} f_{\alpha_n} v^+ \in V_1 \oplus V_2$ if $s = 0$ or $t = 0$), $f_{\beta_1} \dots f_{\beta_m} v^+ \in V_1 \oplus V_2$ (since the only short roots are the $\alpha_i + \dots + \alpha_n$), contrary to our choice of $f_{\beta_1} \dots f_{\beta_m} v^+$. So the claim is proved. ☐

Finally, it is clear that $f_{\alpha_i+\dots+\alpha_n} v^+ \in L(D_n) v^+ \oplus L(D_n) f_{\alpha_n} v^+$ if and only if $f_{\alpha_i+\dots+\alpha_n} v^+ \in L(D_n) f_{\alpha_n} v^+$, by considering the roots that appear in each summand; a similar argument holds for $f_{\alpha_i+\dots+\alpha_n} f_{\alpha_n} v^+$. Putting this together with the claim, we have the lemma. ☐

Assume now that V is the direct sum of two irreducible $L(D_n)$-modules. The group $X = D_n$ contains the full torus T of B_n, so a B_n-high weight vector $v^+ \in V$ must also be a weight vector for D_n. If $v^+ = v_1 + v_2$ ($v_i \in V_i$), then v_1, v_2 are both weight vectors of weight λ, the B_n-high weight. But the high weight space in V has dimension 1. So $v_1 = 0$ or $v_2 = 0$, which implies that v^+ is in one of the summands V_1, V_2. Fix a Borel subgroup B' of D_n contained in the Borel subgroup B of B_n. Since B stabilizes Kv^+, so does B'. Thus v^+ is a maximal vector in one of the irreducible summands. So one of the summands has marking



Call the D_n-module with this marking V_1. Since t interchanges the two summands and corresponds to the graph automorphism, the other summand has marking



We call this D_n-module V_2. Notice another expression for this high weight: $a_1\lambda_1 + \dots + a_{n-2}\lambda_{n-2} + (a_{n-1} + a_n)(\lambda_{n-1} - \lambda_n) + a_{n-1}\lambda_n = a_1\lambda_1 + \dots + a_{n-2}\lambda_{n-2} + (a_{n-1} + a_n)\lambda_{n-1} - a_n\lambda_n = \lambda - a_n\alpha_n$.

Since the B_n-high weight space has dimension 1, $a_n \neq 0$ (else there is a B_n-high weight vector in each summand, which is impossible as above). In the second summand, the high weight is $a_1\lambda_1 + \dots + a_{n-2}\lambda_{n-2} + (a_{n-1} + a_n)\lambda_{n-1} - a_n\lambda_n = \lambda - a_n\alpha_n$.

The weight $\lambda - \alpha_n$ appears in V (since $a_n \neq 0$). So it must appear in one of the D_n-summands (the earlier argument that v^+ is in one of the summands shows that any weight which appears in V must appear in at least one of the summands). It clearly cannot appear in V_1 (the only weights appearing there are $\lambda - \beta$ for $\beta \in \langle \alpha_1, \dots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n \rangle$), so it must be in V_2, which implies $a_n = 1$.

Note that $V_2 = L(D_n) f_{\alpha_n} v^+$ and $V_1 = L(D_n) v^+$. So by the last Lemma,

$$f_{\alpha_i+\dots+\alpha_n} v^+ \in L(D_n) f_{\alpha_n} v^+$$

for all $1 \leq i \leq n$.

For $w \in V$, let $L(D_n)^- w$ denote the span of all $f_{\beta_1} \dots f_{\beta_m} w$ for $\beta_i \in \Sigma^+(D_n)$. Notice that for $w = v^+$ or $w = f_{\alpha_n} v^+$, we have $L(D_n)^- w = L(D_n) w$, as the e_β for $\beta \in \Sigma^+(D_n)$ annihilate these vectors.

Lemma 3.5. *Let $V_{r,k}$ be as before. Fix i and m such that $1 \leq i < m \leq n$. Then $f_{\alpha_r+\dots+\alpha_m} v^+ \in V_{r,m}$ for all $r \in [i, m)$ if and only if $f_{\alpha_r+\dots+\alpha_m} v^+ \in L(D_n)^- f_{\alpha_m} v^+$ for all $r \in [i, m)$.*

Proof. $\Leftarrow$: This is clear, as $L(D_n)^- f_{\alpha_m} v^+ \cap V_{\lambda - (\alpha_r + \dots + \alpha_m)} \subseteq V_{r,m}$.

$\Rightarrow$: This is clear for $m = i + 1$, so assume $s = m - i > 1$ and we have proved the Lemma for $m - i < s$; and that $f_{\alpha_r+\dots+\alpha_m} v^+ \in V_{r,m}$ for all $i \leq r < m$. Consider r such that $i \leq r < m$. If $i < r$ then by the inductive hypothesis $f_{\alpha_r+\dots+\alpha_m} v^+ \in L(D_n)^- f_{\alpha_m} v^+$, and $f_{\alpha_i+\dots+\alpha_m} v^+$ is a linear combination of terms of the form $\dots f_{\alpha_r+\dots+\alpha_m}v^+$, where $r > i$. All these terms are in $L(D_n)^- f_{\alpha_m}v^+$ by induction; thus so is $f_{\alpha_i+\dots+\alpha_m}v^+$. ☐

Now by Proposition 3.1 and the last two Lemmas, $f_{\alpha_i+\dots+\alpha_j}v^+ \in V_{i,j}$ whenever a_i, a_j are consecutive non-zero labels: Lemma 3.4 implies that $f_{\alpha_i+\dots+\alpha_n}v^+ \in L(D_n)f_{\alpha_n}v^+$ for every i; then Lemma 3.5 gives $f_{\alpha_r+\dots+\alpha_n}v^+ \in V_{r,n}$ for every i and every $i \le r < n$; finally, Proposition 3.1 implies that $f_{\alpha_i+\dots+\alpha_j}v^+ \in V_{i,j}$ for every i, where a_j is the first non-zero coefficient after a_i. Recall that $a_n = 1$, and that "$\equiv$" means congruent modulo p.

Lemma 3.6. *Let $1 \le i < j \le n$ such that $a_l = 0$ for $i < l < j$, but $a_i \ne 0 \ne a_j$. Then*

$$f_{\alpha_i+\dots+\alpha_j}v^+ \in V_{i,j} \iff \begin{cases} 2a_i \equiv -2(j-i)-1 & \textit{if } j = n \\ a_i + a_j \equiv i - j & \textit{if } j < n. \end{cases}$$

Proof. The $\lambda - (\alpha_i + \dots + \alpha_j)$-weight space is spanned by

$$\{F_k v^+ = f_{\alpha_i+\dots+\alpha_k} f_{\alpha_{k+1}+\dots+\alpha_j}v^+ | i \le k < j\} \cup \{f_{\alpha_i+\dots+\alpha_j}v^+\}.$$

So $f_{\alpha_i+\dots+\alpha_j}v^+ \in V_{i,j}$ if and only if there is a relation

$$0 = f_{\alpha_i+\dots+\alpha_j}v^+ + \sum_{k=i}^{j-1} b_k F_k v^+. \qquad (**)$$

The vector on the right hand side of $(**)$ is 0 if and only if it is killed by all e_{α_l}'s:

$$\begin{aligned}
0 &= e_{\alpha_j}(f_{\alpha_i+\dots+\alpha_j}v^+ + \textstyle\sum_{k=i}^{j-1} b_k F_k v^+) \\
&= a f_{\alpha_i+\dots+\alpha_{j-1}}v^+ + a_j b_{j-1} f_{\alpha_i+\dots+\alpha_{j-1}}v^+ \quad \Big(a = \\
0 &= e_{\alpha_{j-1}}(f_{\alpha_i+\dots+\alpha_j}v^+ + \textstyle\sum_{k=i}^{j-1} b_k F_k v^+) \\
&= -b_{j-2} f_{\alpha_i+\dots+\alpha_{j-2}} f_{\alpha_j}v^+ + b_{j-1} f_{\alpha_i+\dots+\alpha_{j-2}} \\
0 &= e_{\alpha_{j-2}}(f_{\alpha_i+\dots+\alpha_j}v^+ + \textstyle\sum_{k=i}^{j-1} b_k F_k v^+) \\
&= -b_{j-3} f_{\alpha_i+\dots+\alpha_{j-3}} f_{\alpha_{j-1}+\alpha_j}v^+ + b_{j-2} \\
&\quad \vdots \\
0 &= e_{\alpha_{i+1}}(f_{\alpha_i+\dots+\alpha_j}v^+ + \textstyle\sum_{k=i}^{j-1} b_k F_k v^+) \\
&= -b_i f_{\alpha_i} f_{\alpha_{i+2}+\dots+\alpha_j}v^+ + b_{i+1} f_{\alpha_i} f_{\alpha_{i+2}+\dots+\alpha_j}v^+ \\
0 &= e_{\alpha_i}(f_{\alpha_i+\dots+\alpha_j}v^+ + \textstyle\sum_{k=i}^{j-1} b_k F_k v^+) \\
&= -f_{\alpha_{i+1}+\dots+\alpha_j}v^+ + (a_i+1) b_i f_{\alpha_{i+1}+\dots+\alpha_j}v^+ + \\
&\qquad +b_{j-1} f_{\alpha_{i+1}+\dots+\alpha_j}v^+
\end{aligned}$$

(This uses the fact that we may choose the sign of $N(\alpha, \beta)$ for all extraspecial pairs α, β ([1, page 58]). We choose them so that $[e_{\alpha_{k+1}+\dots+\alpha_j}, e_{\alpha_k}] = -e_{\alpha_k+\dots+\alpha_j}$; this forces all of the above signs.)

For $l < i$ or $l > j$, $e_{\alpha_l}(f_{\alpha_i+\dots+\alpha_j}v^+ + \sum_{k=i}^{j-1} b_k F_k v^+) = 0$ trivially.

So if $j = n$, the above implies that $b_i = \dots = b_{j-1} = \dfrac{-2}{a_j} = -2$. Then the last equation (for e_{α_i}) gives the relation $-1 - 2(a_i+1) + (j-i-1)(-2) = 0$; i.e. $-2a_i - 2(j-i) - 1 = 0$, or $2a_i = -2(j-i) - 1$.

If $j < n$, then $b_i = \dots = b_{j-1} = \dfrac{-1}{a_j}$. So the equation for e_{α_i} gives $-1 - \frac{1}{a_j}(a_i+1) - \frac{1}{a_j}(j-i-1) = 0$, or $a_j + a_i = i - j$.

This completes the proof of the Lemma. ☐

That the set of congruences given in the Lemma is equivalent to the conclusion of Theorem 3.3 is clear: If a_k is the last non-zero label before a_n, then

$$\begin{aligned}
1 &\equiv \langle \lambda + \rho, \alpha_k + \dots + \alpha_n \rangle = 2\langle \lambda + \rho, \alpha_k + \dots + \alpha_{n-1} \rangle + \langle \lambda + \rho, \alpha_n \rangle = 2(a_k+1) + 2(n-k-1) + 2 \\
&\Leftrightarrow 2a_k \equiv -2(n-k) - 1.
\end{aligned}$$

Also, if a_i, a_j are consecutive non-zero labels with $j < n$, then

$$\begin{aligned} 1 &\equiv \langle \lambda+\rho, \alpha_i+\cdots+\alpha_j\rangle = (\lambda+\rho, \alpha_i+\cdots+\alpha_j) = a_i + a_j + (j-i+1) \\ \Leftrightarrow &\quad a_i + a_j \equiv i - j. \end{aligned}$$

This completes the proof of Theorem 3.3 in one direction.

Now assume V is as in the setup of the theorem (an irreducible restricted KB_n-module), and that λ satisfies the congruences in the last Lemma (by the comments above, this is equivalent to satisfying the congruences in the Theorem). We need to show V is irreducible as a $\langle D_n, t\rangle$-module.

By Lemmas 3.4 and 3.5, if we show $f_{\alpha_i+\cdots+\alpha_n}v^+ \in V_{i,n}$ and $f_{\alpha_i+\cdots+\alpha_n} f_{\alpha_n} v^+ \in L(D_n)v^+$ for all $i < n$, then $V = L(D_n)v^+ \oplus L(D_n) f_{\alpha_n} v^+$ as an $L(D_n)$-module. First we show that in fact $f_{\alpha_i+\cdots+\alpha_m}v^+ \in V_{i,m}$ for every i and m with $1 \le i < m \le n$:

a) Assume there are no non-zero labels between a_i and a_m. Then by Lemma 3.6, $f_{\alpha_i+\cdots+\alpha_m}v^+ \in V_{i,m}$. there is a relation $f_{\alpha_i+\cdots+\alpha_n}v^+ = -\sum_{k=i}^{n-1} b_k F_k v^+ \in L(D_n) f_{\alpha_n} v^+ \subseteq V_{i,n}$.

b) Assume there are $r > 2$ non-zero labels between a_i and a_m (inclusive), and that we have proved the result for fewer than r non-zero labels. Then

$$f_{\alpha_i+\cdots+\alpha_m}v^+ = \pm\big(f_{\alpha_i+\cdots+\alpha_{m-1}} f_{\alpha_m} v^+ - f_{\alpha_m} f_{\alpha_i+\cdots+\alpha_{m-1}} v^+\big).$$

We have $f_{\alpha_i+\cdots+\alpha_{m-1}} f_{\alpha_m} v^+ \in V_{i,m}$, and $f_{\alpha_i+\cdots+\alpha_{m-1}}v^+ \in V_{i,m-1}$ by induction. So $f_{\alpha_i+\cdots+\alpha_{m-1}}v^+$ is a sum of terms of type (†) with more than one f_β. Then f_{α_m} commutes with all but the last f_β in each of the terms; thus $f_{\alpha_m} f_{\alpha_i+\cdots+\alpha_{m-1}} v^+ \in V_{i,m}$. So $f_{\alpha_i+\cdots+\alpha_m}v^+ \in V_{i,m}$.

Next we must show that $f_{\alpha_i+\cdots+\alpha_n} f_{\alpha_n} v^+ \in L(D_n)v^+$ for every $i < n$. We use induction, doing the base cases much as in the proof of Lemma 3.6.

Let $i = n-1$. The $\lambda - (\alpha_{n-1} + 2\alpha_n)$-weight space is spanned by

$$\{f_{\alpha_{n-1}+2\alpha_n}v^+, f_{\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+, f_{\alpha_{n-1}} f_{\alpha_n} f_{\alpha_n} v^+\}.$$

The third element in this set is 0, as $a_n = 1$. Set $b = 1$ if $a_{n-1} = 0$, and set $b = -2$ if $a_{n-1} \neq 0$. Then the vector

$$w = f_{\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+ + b f_{\alpha_{n-1}+2\alpha_n} v^+$$

is trivially annihilated by e_{α_l} for $l < n-1$; each summand is annihilated by $e_{\alpha_{n-1}}$; and finally,

$$\begin{aligned} &e_{\alpha_n} f_{\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+ + b e_{\alpha_n} f_{\alpha_{n-1}+2\alpha_n} v^+ \\ = &\quad f_{\alpha_{n-1}+\alpha_n} h_{\alpha_n} v^+ + 2 f_{\alpha_{n-1}} f_{\alpha_n} v^+ + b f_{\alpha_{n-1}+\alpha_n} v^+ \\ = &\quad \begin{cases} (1 + 2N(-\alpha_{n-1}, -\alpha_n) + b) f_{\alpha_{n-1}+\alpha_n} v^+ = (b-1) f_{\alpha_{n-1}+\alpha_n} v^+ = 0 & \text{if } a_{n-1} = 0, \\ (b+2) f_{\alpha_{n-1}+\alpha_n} v^+ = 0 & \text{if } a_{n-1} \neq 0. \end{cases} \end{aligned}$$

Since w is not a high weight vector, this shows that it must in fact be 0, so $f_{\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+ \in L(D_n)v^+$.

Next let $i = n-2$. Now the $\lambda - (\alpha_{n-2} + \alpha_{n-1} + 2\alpha_n)$-weight space is spanned by

$$\{f_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n}v^+, f_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+, f_{\alpha_{n-2}} f_{\alpha_{n-1}+2\alpha_n} v^+\},$$

as we just showed that we needn't include $f_{\alpha_{n-2}} f_{\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+$, and

$$f_{\alpha_{n-2}+\alpha_{n-1}} f_{\alpha_n} f_{\alpha_n} v^+ = 0 = f_{\alpha_{n-2}} f_{\alpha_{n-1}} f_{\alpha_n} f_{\alpha_n} v^+,$$

since $a_n = 1$.

If $a_{n-1} = 0$, let $b_1 = 2$ and $b_2 = -2$. If $a_{n-1} \neq 0$, let $b_1 = 2$ and $b_2 = 1$. Then in calculations similar to those above, we find that e_{α_l} annihilates

$$w = f_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+ + b_1 f_{\alpha_{n-2}} f_{\alpha_{n-1}+2\alpha_n} v^+ + b_2 f_{\alpha_{n-2}+\alpha_{n-1}+2\alpha_n} v^+,$$

so in fact this sum is 0 and $f_{\alpha_{n-2}+\alpha_{n-1}+\alpha_n} f_{\alpha_n} v^+ \in L(D_n)v^+$ (the only complication is that in the case $a_{n-1} \neq 0$, we were forced to again apply the e_{α_l} to show that $e_{\alpha_n} w = 0$).

Finally, assume that $i \le n-3$. Then

$$f_{\alpha_i+\cdots+\alpha_n} f_{\alpha_n} v^+ = f_{\alpha_n} f_{\alpha_i+\cdots+\alpha_n} v^+ + b f_{\alpha_i+\cdots+2\alpha_n} v^+$$

for some b, so $f_{\alpha_i+\cdots+\alpha_n} f_{\alpha_n} v^+ \in L(D_n)v^+$ if and only if $f_{\alpha_n} f_{\alpha_i+\cdots+\alpha_n} v^+ \in L(D_n)v^+$. We have

$$\begin{aligned} & f_{\alpha_n} f_{\alpha_i+\cdots+\alpha_n} v^+ \\ = \quad & \pm(f_{\alpha_n} f_{\alpha_i+\cdots+\alpha_{n-2}} f_{\alpha_{n-1}+\alpha_n} v^+ - f_{\alpha_n} f_{\alpha_{n-1}+\alpha_n} f_{\alpha_i+\cdots+\alpha_{n-2}} v^+) \\ = \quad & \pm(f_{\alpha_i+\cdots+\alpha_{n-2}} f_{\alpha_n} f_{\alpha_{n-1}+\alpha_n} v^+ - f_{\alpha_n} f_{\alpha_{n-1}+\alpha_n} w), \end{aligned}$$

where $w \in V_{i,n-2}$ (we have $f_{\alpha_\ell+\cdots+\alpha_m} \in V_{\ell,m}$ whenever $1 \le \ell < m \le n$). The first term in this last expression is in $L(D_n)v^+$ by the above, and w is a sum of terms of type (†) with at least two f_β. Thus $f_{\alpha_n} f_{\alpha_{n-1}+\alpha_n}$ commutes with all but the last of these f_β, and by induction we are again in $L(D_n)v^+$. So $f_{\alpha_i+\cdots+\alpha_n} f_{\alpha_n} v^+ \in L(D_n)v^+$ for every $i < n$.

We still must show that $V_1 = L(D_n)v^+$ and $V_2 = L(D_n) f_{\alpha_n} v^+$ are irreducible $L(D_n)$-modules. By the above, $V = V_1 \oplus V_2$ as $L(D_n)$-modules. Also, $V \cong V^*$ as $L(B_n)$-modules, thus as $L(D_n)$-modules. Now consider V as an $L(D_n)$-module; it has a quotient $V(\lambda)$. If n is even, then $V(\lambda)^* \cong V(\lambda)$ ([10, 1.8]), so $V^* \cong V$ has $V(\lambda)$ as a submodule. Since v^+ is the only vector of weight λ, and $L(D_n)v^+ = V_1$, we must have $V_1 \cong V(\lambda)$. If n is odd, then $V(\lambda)^* \cong V(\lambda-\alpha_n)$, so V has $V(\lambda-\alpha_n)$ as a submodule. Since $f_{\alpha_n}v^+$ spans $V_{\lambda-\alpha_n}$, we have $V_2 = L(D_n) f_{\alpha_n} v^+ \cong V(\lambda-\alpha_n)$.

Now doing the same thing with $V(\lambda-\alpha_n)$ instead of $V(\lambda)$, we see that $V_1 \cong V(\lambda)$ and $V_2 \cong V(\lambda-\alpha_n)$. So V_1, V_2 are irreducible $L(D_n)$-modules.

So the Theorem is proved for $p \neq 2$.

Now assume $p = 2$. By [10, 1.6], $V = V' \otimes V''$ as a KB_n- module (thus as a KD_n-module), where V' is the irreducible module with high weight $a_1\lambda_1 + \cdots + a_{n-1}\lambda_{n-1}$, and V'' has high weight $a_n\lambda_n$. If we assume that V is the sum of two irreducibles for $L(D_n)$ ($V = V_1 \oplus V_2$), then the same argument as before shows $a_n = 1$. Then V'' is the sum of two non-zero irreducibles for $L(D_n)$, say $V'' = W_1 \oplus W_2$. So $V' \otimes V'' = (V' \otimes W_1) \oplus (V' \otimes W_2) = V_1 \oplus V_2$ as D_n-modules. But the same result in [10] says that no restricted irreducible D_n-modules are tensor decomposable. So one of V', V'' must be 0. Since $V'' \neq 0$, we must have $V' = 0$, so $a_1 = a_2 = \cdots = a_{n-1} = 0$.

This completes the proof of Theorem 3.3. □

4. WHEN $W|_X$ IS REDUCIBLE

The notation will be as in the introduction: G is an algebraic group over a field K of arbitrary characteristic p (0 or a prime), with simple identity component X admitting an outer (graph) automorphism. So except for the case $X = D_4$, the Dynkin diagram for X has a single graph automorphism inducing an automorphism t of X, and $G = X\langle t\rangle$. If $X = D_4$, then $\mathrm{Aut}(X) = D_4 \cdot \mathrm{Sym}_3$. Let $\mathrm{Sym}_3 = \langle s,t\rangle$, with $s^3 = 1 = t^2$. Then the possibilities are: $G = D_4\langle t\rangle$; $G = D_4\langle s\rangle$; and $G = D_4\langle s,t\rangle$.

Let Y be a simple algebraic group of classical type and of rank n such that $X \le Y$ and $G \le \mathrm{Aut}(Y)$ (t may or may not be in Y; but if $X = D_4$ and $s \in G$ then $s \in Y$ since no simple group properly containing D_4 has an outer automorphism of order 3). Let $\{\lambda_i\}$ ($\{\delta_i\}$) be the set of fundamental dominant weights of Y (X), $\{\alpha_i\}$ ($\{\beta_i\}$) the set of fundamental roots for Y (X) with respect to some maximal torus $T_Y < Y$ ($T_X < X$) and some Borel subgroup. Let $V(\lambda) = V$ be an irreducible Y-module on which G acts irreducibly but X acts reducibly, with $\lambda = \sum a_i\lambda_i$.

In this section we consider the case when X acts reducibly on the natural module W for Y. The main theorem of the section is the following:

Theorem 4.1. *Assume X acts reducibly on the natural module W for Y. Then either $Y = \mathrm{SL}(V)$, $\mathrm{SO}(V)$, $\mathrm{Sp}(V)$, or (X,Y,V) are as in* U_1, U_2, U_3, *or* U_4 *of Table 2.*

The approach will be to analyze the various possibilities for a minimal X- or G-invariant subspace of W.

4.1. **Preliminary Lemmas.**

Several possible situations for the action of X on W will arise more than once, so we have some Lemmas to prove first.

Lemma 4.2. *Assume P_Y is a maximal parabolic subgroup of Y corresponding to the root α_j, with $j = n$ if Y is not of type A_n. Let $P_Y = L_Y Q_Y$ be a Levi decomposition. If there exist only two non-zero quotients $V^i(Q_Y) = [V, Q_Y^{i-1}]/[V, Q_Y^i]$, and* $\dim(V^1(Q_Y)) = \dim(V^2(Q_Y))$, *then one of the following holds:*

1. *$V \cong W$;*
2. *n is odd, $Y = A_n$, $L'_Y = A_{n-1}$, $\lambda = \lambda_{(n+1)/2}$; or*
3. *$Y = D_4$, $X = A_3$, $\lambda = \lambda_3$, with $V|_X = V_X(\delta_1) \oplus V_X(\delta_3)$ and either $W|_X = V_X(\delta_2) \oplus 2V_X(0)$, or $W|_X = V_X(\delta_1) \oplus V_X(\delta_3)$.*

Proof. Let Λ be the set of T_Y-weights of V. Write $V^i = \sum_{\mu \in \Lambda_i} V_\mu$, where $\Lambda_i = \{\mu \in \Lambda | \mu = \lambda - \beta$, coefficient of α_j in $\beta = i - 1\}$. By 2.7 in [10], $V^1 \cong V/[V, Q_Y] = V^1(Q_Y)$ and $V^2 \cong [V, Q_Y]/[V, Q_Y, Q_Y] = V^2(Q_Y)$, so $\dim(V^1) = \dim(V^2)$.

Since $V = V^1 \oplus V^2$, the weight space $V_{w_0\lambda}$ must be in V^2 (if $V_{w_0\lambda} \subseteq V^1$ then $V^2 = 0$, so $V = V^1 = V/[V, Q_Y]$ — thus $V = V^1$ is an irreducible L_Y-module, which is a contradiction as $L_Y < P_Y$), i.e. $w_0\lambda = \lambda - (\cdots + \alpha_j + \cdots)$. So $\lambda - w_0\lambda$ has α_j-coefficient 1. In the cases where $w_0 = -1$ (D_n for n even, B_n, C_n; here $j = n$), this implies $2\lambda = \cdots + \alpha_n$. In the other cases ($A_n$, D_n for n odd), let σ be an involutory graph automorphism of the group; then the above gives $\lambda + \lambda^\sigma = \cdots + \alpha_j + \ldots$ (with $j = n$ in the D_n case). The cases are:

- $Y = A_n$, $L'_Y = A_{j-1} \times A_{n-j}$: We use the expression for the λ_i in terms of the α_i (e.g. [7, page 69]) to calculate the coefficient (which must be 1 by the above) of α_j in $\lambda + \lambda^\sigma$. By reversing the labelling if necessary, we may assume $j \leq (n+1)/2$. It is a straightforward calculation to see that we obtain $1 = (a_1 + a_n) + 2(a_2 + a_{n-1}) + \cdots + j(a_j + a_{j+1} + \cdots + a_{n-j+1})$. The only nonnegative integral solutions to this equation are $a_1 + a_n = 1, a_i = 0$ for $i \neq 1, n$; and $j = 1, \sum a_i = 1$. The first gives $V \cong W$. So assume $j = 1$, $1 = \sum a_i$. Assume $a_k = 1, a_i = 0$ for $i \neq k$. By assumption, $\dim(V/[V, Q_Y]) = \dim([V, Q_Y]/[V, Q_Y, Q_Y])$.

  By Lemma 2.2, the high weight of $V/[V, Q_Y]$ as an L'_Y-module is $\sum_{i=2}^{n} a_i\lambda_i = \lambda_k|_{L'_Y}$ ($k = 1$ cannot occur because this would give a top quotient of dimension 1 (high weight 0) and λ_2 as a high weight for the bottom quotient (as L'_Y-modules) by Lemma 2.4; but we know the two quotients have the same dimension). The bottom quotient has a high weight $\lambda - (\alpha_1 + \cdots + \alpha_k)|_{L'_Y} = \lambda_{k+1}|_{L'_Y}$, and since for any k the dimension of the Y-module with high weight λ_k ($\binom{n+1}{k}$) is the sum of the dimensions of the L'_Y-modules with high weights $\lambda_k|_{L'_Y}$ and $\lambda_{k+1}|_{L'_Y}$ ($\binom{n}{k-1} + \binom{n}{k}$), λ_{k+1} is in fact the only high weight of the bottom quotient. These two must have the same dimension, so $\binom{n}{k-1} = \binom{n}{k}$; thus n is odd and $k = (n+1)/2$. So this setup does give an example of two modules for A_{n-1} summing to a module for A_n, and we get item (2) of the Lemma.
- $Y = B_n$: Here the coefficient of α_n in 2λ is $2\sum_{i=1}^{n-1} a_i i + a_n n$. This must be 1, which is impossible.
- $Y = C_n$: The coefficient of α_n in 2λ is $\sum_{i=1}^{n} a_i i$. This coefficient must be 1, which implies $a_1 = 1, a_i = 0$ for $i > 1$. So here we have only the case $V \cong W$.
- $Y = D_n$, $j = n$ **odd**: The coefficient of α_n in $\lambda - w_0\lambda = \lambda + \lambda^\sigma$ is

  $$\sum_{i=1}^{n-2} a_i i + (a_{n-1} + a_n)\left(\frac{n-1}{2}\right).$$

  This coefficient must be 1, which implies either $a_1 = 1, a_i = 0$ for $i > 0$, which gives $V \cong W$, or $n = 3, a_1 = 0, a_2 + a_3 = 1$. In the second instance, our assumption is that $\Pi(L_Y) = \{\alpha_1, \alpha_2\}$, so there are two distinct possibilities:

  $a_2 = 1$: In the chain $V > [V, Q_Y] > 0$, the top factor is an irreducible $L'_Y = A_2$-module with high weight $\lambda_2|_{L'_Y}$; the second factor has high weight $\lambda - (\alpha_1 + \alpha_2 + \alpha_3)|_{L'_Y} = 0$. These two have different dimensions, so we get no examples here.

  $a_3 = 1$: As above, the top factor has high weight 0 for L'_Y; the second factor has high weight $\lambda - \alpha_3 = \lambda_1|_{L'_Y}$. Again, these do not have the same dimension.

- $Y = D_n$, $n \geq 4$ **even**: The coefficient of α_n in 2λ is
$$\sum_{i=1}^{n-2} a_i i + a_{n-1}\left(\frac{n-2}{2}\right) + a_n\left(\frac{n}{2}\right).$$
This must be 1, which implies either $a_1 = 1, a_i = 0$ for $i > 1$, which gives $V \cong W$; or $n = 4, \lambda = \lambda_3$, which gives $\dim(V) = 8$. Since X is proper in $Y = D_4$, we have $X \neq D_4$ and hence $G = X\langle t\rangle$. So as X-modules, $V = V_1 \oplus V_2$, with V_1 and V_2 irreducible X-modules, interchanged by t.

  We have $\dim(V_1) = 4$, and the only simple groups of rank 4 or less with an irreducible module of dimension 4 are A_1 C_2, and A_3. So X is of type A_3 (not A_1 or C_2 because X admits a graph automorphism), and V_1 has high weight δ_1 or δ_3.

  Every A_3-module of dimension $\leq 6p$ is completely reducible by [8], so W is completely reducible as an X-module. Let δ be a T_X-high weight of W. If δ is not restricted, then by Steinberg's tensor product theorem ([13]), $V_X(\delta) = W_1^{q_1} \otimes \cdots \otimes W_r^{q_r}$, where the W_i are restricted irreducible X-modules and the q_i are distinct powers of p. If there are two or more terms in this product, then $\dim V_X(\delta) > 8$ (as each W_i has dimension at least 4), so there is in fact only one term, $W_1^{q_1}$. But then, as in the proof of Lemma 2.6, $q_1 = 1$, as there is no twist in the action of X on V (which factors through the embedding of X in Y).

  We must have δ_2 or both δ_1 and δ_3 as X-high weights of W (since the only A_3-high weight modules of dimension 8 or less are those with high weight 0, δ_1, δ_2, or δ_3; X acts nontrivially on W, ruling out 0; and t does act on W, so if δ_1 appears then so does δ_3). In fact, both of these possibilities give the other examples listed in the Lemma: Let X be the derived group of the Levi factor of the parabolic subgroup corresponding to $\{\alpha_2, \alpha_3, \alpha_4\} \subseteq \Pi(Y)$. Then X is of type $D_3 \cong A_3$; W restricts to X as $V_{A_3}(\delta_2) \oplus 2V_{A_3}(0)$, and the Y-module $V = V_{D_4}(\lambda_3)$ restricts to X as $V_{A_3}(\delta_1) \oplus V_{A_3}(\delta_3)$. Similarly, if we let X be the derived group of the Levi factor of the parabolic subgroup corresponding to $\{\alpha_1, \alpha_2, \alpha_3\}$, then W restricts to X as $V_{A_3}(\delta_1) \oplus V_{A_3}(\delta_3)$, as does V. In this latter case $V \cong W$ as X-modules, but not as Y-modules.

  Finally, in the two cases above there is an element $t \in Y$ acting as a graph automorphism on the specified $A_3 \leq Y$. Consider the first case: X corresponds to the subsystem $\{\alpha_2, \alpha_3, \alpha_4\} \subseteq \Pi(Y)$. Here V restricts to X as $V_{A_3}(\delta_1) \oplus V_{A_3}(\delta_3) = U \oplus U^*$, where U is the natural module for X. Let $\{u_1, u_2, u_3, u_4\}$ be a basis for $U \leq V$, and $\{u_i^*\}$ a basis for U^*. Then by Witt's theorem, there is an element of $O(V)$ sending u_i to u_i^* and u_i^* to u_i for every i; this element is in fact of determinant 1 (it has determinant 1 on every 2-space $\langle u_i, u_i^*\rangle$), so is in $\mathrm{SO}(V) = Y$.

  Similarly consider the second of the possible embeddings $X \hookrightarrow Y$: X corresponds to the subsystem $\{\alpha_1, \alpha_2, \alpha_3\} \subseteq \Pi(Y)$. Here W restricts to X as the sum of the natural module and its dual, and we proceed as above to find $t \in Y$.

□

Recall that $V|_X = V_1 \oplus \cdots \oplus V_k$, with V_i irreducible ($k = 2$ except possibly when $X = D_4$).

Lemma 4.3. *Assume that $W|_X = W_1 \oplus \cdots \oplus W_l$ where each W_i is irreducible as an X-module;* $\mathrm{rad}(W_i) = 0$ *(so Y has type B_n, C_n, or D_n); and G permutes the W_i transitively. If $X < I(W_1)' \times \cdots \times I(W_l)' < Y$ and $I(W_i)'$ acts trivially on every V_μ on which $I(W_j)'$ (for some $j \neq i$) acts nontrivially, then $V \cong W$.*

Proof. Let $Y_i = I(W_i)'$. When we say that Y_i acts trivially on V_μ, we mean that $1 \times \cdots \times Y_i \times 1 \times \cdots \times 1$ acts as scalars on V_μ. Since G permutes the Y_i transitively, $Y_i \cong Y_j$. The natural module W is symplectic or orthogonal and $\mathrm{rad}(W_i) = 0$, so each Y_i is also symplectic or orthogonal. So the possibilities are $X \hookrightarrow Y_i$ irreducible on V_i for every i, with X proper in each Y_i; and $X \cong Y_i$ for every i. Examining Table 1 in [10], we see that there are no possibilities for the first option except perhaps $V_i =$ natural module for Y_i for every i. But the natural modules for the Y_i are the W_i, and they sum to the natural module for Y. So this gives $V \cong W$.

If $X \cong Y_i$, then since X has a graph automorphism, we have $X \cong Y_i = D_m$ (not A_n because W_i is orthogonal or symplectic) for some m; $Y = D_{lm}$ if $p \neq 2$; $Y = Dlm$ or C_{lm} if $p = 2$.

Combine orthogonal bases $\{w_1^{(i)}, w_{-1}^{(i)}, \dots, w_m^{(i)}, w_{-m}^{(i)}\}$ for the W_i to get an orthogonal basis for W. Write $W_i^+ = \langle w_1^{(i)}, \dots, w_m^{(i)}\rangle$; then let $P_Y = \mathrm{stab}(W_1^+) \times \mathrm{stab}(W_2^+) \times \cdots \times \mathrm{stab}(W_{l-1}^+) \times \mathrm{stab}(W_l)$. With respect to an appropriate Borel subgroup and maximal torus, P_Y is the parabolic subgroup of Y corresponding to $\Pi(Y) - \{\alpha_m, \alpha_{2m}, \dots, \alpha_{(l-1)m}\}$; and each simple factor of L_Y' ($P_Y = L_Y Q_Y$) is contained in a distinct Y_i.

Recall that $\lambda = \sum a_i \lambda_i$ is the T_Y-high weight for V. As Y_i acts trivially where Y_j acts nontrivially, we may conclude that only one factor of L_Y' acts nontrivially on the highest weight space V_λ of the irreducible L_Y'-module $V/[V,Q]$. If $a_{jm} \neq 0$ for some $j < l$, then $\lambda - \alpha_{jm}$ is an L_Y'-high weight in $[V,Q_Y]/[V,Q_Y,Q_Y]$, and has non-zero labels on two connected components of $\Pi(L_Y')$, so as above we can conclude that in fact $a_{jm} = 0$ for every $j < l$.

Let $j > m$ be minimal with respect to $a_j \neq 0$. By the above, $m \nmid j$ or $j = n$. Then $j = am + r$ for some a, with $0 \leq r < m$. If $r \neq m-1$, then $\lambda - (\alpha_{am} + \cdots + \alpha_j)$ is a L_Y'-high weight in $V^2(Q_Y)$ with non-zero labels on two connected components, which is a contradiction. If $r = m-1$ and $j \neq n-1$, then $\lambda - (\alpha_j + \cdots + \alpha_{(a+1)m})$ is the required high weight of $V^2(Q_Y)$. Finally, if $j = n-1$ or $j = n$, then $\lambda - (\alpha_{(l-1)m} + \cdots + \alpha_j)$ (if $Y = C_n$) or $\lambda - (\alpha_{(l-1)m} + \cdots + \alpha_{n-2} + \alpha_j)$ is a high weight giving the same contradiction. So $a_j = 0$ for $j \geq m$.

If $a_i \neq 0$ for some $1 < i < m$ or $a_1 > 1$ (let $i = 1$ in this case), with $a_j = 0$ for $j > i$, then $\lambda - (\lambda_i + \cdots + \lambda_m)$ is an L_Y'-high weight in $[V,Q_Y]/[V,Q_Y,Q_Y]$, again with non-zero labels on two connected components of $\Pi(L_Y')$. So the only possibility is $a_1 = 1, a_i = 0$ for $i > 1$; in other words, $V \cong W$. □

Lemma 4.4. *Assume that* $G = X\langle t\rangle$*, and that* $W = W_1 \oplus W_2$*, with* X *stabilizing* W_1 *and* W_2*,* t *stabilizing the decomposition, and* $\operatorname{rad}(W_1) = 0$*. Then one of the following holds:*

1. $V \cong W$*;*
2. U_2 *of Table 2; or*
3. U_3 *of Table 2.*

Proof. Let $Y_1 = I(W_1)'$; $Y_2 = I(W_2)'$. Then $X < Y_1 \times Y_2$, with t either interchanging the two factors or stabilizing each.

The cases are: **A.** V is irreducible for $Y_1 \times Y_2$; **B.** V is reducible for $Y_1 \times Y_2$.

A) We can read off the possibilities for V and $Y_1 \times Y_2 < Y$ from Table 1 in [10]:

1. (IV$_2$, IV$_2'$, S$_6$ in [10]): $Y = D_{n+1}, \lambda = \lambda_n$ or $\lambda_{n+1}, Y_1 \times Y_2 = B_{n-k} \times B_k$, $V = V(\lambda), V|_{Y_1 \times Y_2} = V(\varepsilon_{n-k}) \otimes V(\varepsilon_k')$, where $\{\varepsilon_i\}$ and $\{\varepsilon_i'\}$ are sets of fundamental dominant weights for Y_1 and Y_2; *OR*

(MR$_5$ in [10]): $p = 2$, $Y = B_n$, $\lambda = \lambda_n$, $Y_1 \times Y_2 = B_k \times B_{n-k}$, $V = V(\lambda)$, $V|_{Y_1 \times Y_2} = V(\varepsilon_k) \otimes V(\varepsilon_{n-k}')$ (ε_i, ε_i' as above).

In both these cases $V = V' \otimes V'' = V_1 \oplus V_2$ as X-modules, for some X-modules V' and V''. If V' or V'' is irreducible as an X-module, we can check Table 1 in [10] and see that there are no simple connected proper subgroups of B_{n-k} (B_k) which act irreducibly on V' (V''). This would force $X \cong B_{n-k}$ (B_k), which has is impossible because X admits a graph automorphism. So V', V'' are both reducible as X-modules; let $V_1' < V'$ and $V_1'' < V''$ be proper non-zero X-invariant subspaces. Then $V_1' \otimes V_1'', V_1' \otimes V''$, and $V' \otimes V_1''$ are all proper non-zero X-invariant subspaces of V. But there are only two such X-invariant subspaces. So we have no examples here.

2. (IV$_1$, IV$_1'$ in [10]): $Y = D_{n+1}, Y_1 \times Y_2 = 1 \times B_n, \lambda = a\lambda_n + b\lambda_i$ or $a\lambda_{n+1} + b\lambda_i$, either $b = 0$ or ($a \neq 0 \neq b$ and $a + b + n - i \equiv 0 \pmod{p}$), $V|_{Y_2} = V(a\varepsilon_n + b\varepsilon_i)$ (with $\{\varepsilon_i\}$ the set of fundamental dominant weights for Y_2).

Here W_1 has dimension 1, and so t stabilizes W_1 and W_2 (we know it stabilizes the decomposition $W = W_1 \oplus W_2$, and if it interchanged them we would have $Y = A_1$, which is impossible). So we have the situation $X \leq B_n$, $G \leq \operatorname{Aut}(B_n)$, with V an irreducible B_n- and G-module but reducible X-module. So we may use induction. In [5], it is shown that there are no examples with X acting irreducibly on the natural module for B_n. So X must act reducibly on the natural module for B_n; i.e. we must be back in the situation we investigate in this section. So by induction, $X = D_n$ (the only case in the Theorem where B_n appears as an overgroup is for $X = D_n$).

In Theorem 3.3 we saw that $V = V(\lambda)$ is irreducible for $D_n\langle t\rangle < B_n$ if and only if $a = 1$ and $b = 0$, or $a = 1$ and $2b \equiv -2(n-i) - 1 \pmod{p}$. But $2b + 2n - 2i + 1 \equiv 0$ and $b + n - i + 1 \equiv 0$ imply that $1 \equiv 0 \pmod{p}$. So $b = 0$ gives the only examples here; this is U_3 of Table 2.

B) V is reducible for $Y_1 \times Y_2$. Recall that t normalizes $Y_1 \times Y_2$.

Let V' be a minimal proper $Y_1 \times Y_2$-invariant subspace of V. Since $X < Y_1 \times Y_2$, $V' = V_1$ or V_2; without loss of generality assume $V' = V_2$. The product $Y_1 \times Y_2$ acts irreducibly on V_2, so $V_2 = V_2' \otimes V_2''$ as $Y_1 \times Y_2$-modules, for some Y_1-representation V_2' and Y_2-representation V_2''. By Lemma 2.1, one of V_2', V_2'' must be trivial (since no restricted irreducible X-modules are tensor decomposable).

So one of Y_1, Y_2 acts irreducibly on V_2 while the other acts trivially. Without loss of generality assume Y_2 is irreducible on V_2 and Y_1 acts trivially.

By our assumption that t stabilizes the decomposition $W = W_1 \oplus W_2$, t normalizes $Y_1 \times Y_2$ and interchanges V_1 and V_2. So, as above, one of Y_1, Y_2 acts irreducibly on V_1 and the other acts trivially. In particular, if t interchanges Y_1 and Y_2, then Y_1 is irreducible on V_1 and Y_2 on V_2; if t preserves each of the factors Y_1 and Y_2, then Y_2 is irreducible on both V_1 and V_2.

Assume Y_2 is irreducible on both V_1 and V_2. Then Y is irreducible on $V = V_1 \oplus V_2$ and $Y_1 < Y$ acts trivially; thus $Y_1 < Z(Y)$. But Y is simple, so $Z(Y)$ is finite. So $Y_1 = 1 = I(W_1)'$; so $\dim(W_1) \leq 2$ and W is orthogonal, since $\mathrm{rad}(W_1) = 0$. Then $X \leq Y_2 < Y$ with $(Y_2, Y) = (B_{n-1}, D_n)$, (D_n, B_n), (D_{n-1}, D_n), or (B_{n-1}, B_n), with V_1 and V_2 both irreducible Y_2-modules. In the first and last cases ($Y_2 = B_{n-1}$), we have $X \neq Y_2$ (since X admits a graph automorphism), so either V_i is the natural module for Y_2, or the triple (X, Y_2, V_i) appears in Table 1 of [10]. No such triples ($Y_2 = B_{n-1}$, X admitting a graph automorphism, $V_i|_X$ not symmetric with respect to the graph automorphism) appear in that table, so V_i must be the natural module for Y_2, of dimension $2n-1$. Then $\dim V = 4n-2$. But $Y = D_n$ or B_n, and neither of these groups have any restricted irreducible modules of dimension $4n-2$ in any characteristic. So these cases don't occur.

If $(Y_2, Y) = (D_{n-1}, D_n)$, then the same arguments as above apply if $X \neq D_{n-1}$, with the exception that now $\dim V = 4n-4$, and D_5 does have an irreducible restricted module of dimension 16. So $X = A_2$ or A_3, and we need to check for irreducible X-modules of dimension 8, whose high weights are not symmetric with respect to the graph automorphism. There are no such modules.

So $X = D_{n-1}$. Consider a group of type B_{n-1} sitting between D_{n-1} and D_n. If $V|_{B_{n-1}}$ is irreducible, then as at the end of item 2) of case A) above, we are in case U_3 of Table 2. If $V|_{B_{n-1}}$ is reducible, let V' be a minimal B_{n-1}-invariant subspace. Since $X < B_{n-1}$, $V' = V_1$ or V_2, say V_1. As above, we check Table 1 of [10] and find that V_1 must be the natural module for B_{n-1}. But $\dim V_1 = \dim V_2$, so $\dim V = 4n-2$, which is impossible as above.

Finally, if $X \leq D_n < B_n$, we know by Theorem 3.3 what $V|_Y, V|_{Y_2}$ are. By examining Table 1 in [10], we see that there is only one proper connected simple subgroup $X < D_n$ admitting a graph automorphism which is irreducible on any of the D_n-modules we obtain in Theorem 3.3; this occurs in S_7 of the table. But in this case, $(V_1|_X)^t \cong V_1|_X$, so we have no example So $X = D_n$ is the only case here; this is U_2 of Table 2.

So we are left with Y_1 acting irreducibly on V_1 and Y_2 on V_2; and since t interchanges them, $Y_1 \cong Y_2$. But now we are exactly in the situation of Lemma 4.3, which tells us that $V \cong W$. □

4.2. **Proof of Theorem 4.1 for $G = X\langle t \rangle$.** Throughout this section we assume that $G = X\langle t \rangle$, where t is an involutory graph automorphism of X.

If t does not act on W, then (since $t \in \mathrm{Aut}(Y)$) we must have $Y = A_n$ (if $Y = D_4$, we have chosen W to be the 8-dimensional module fixed by t), and we let A be a minimal X-invariant subspace of W. If G acts irreducibly on W, let $A < W$ be a minimal X-invariant subspace (so $W = \langle G \cdot A \rangle$). If G acts reducibly on W, let $A < W$ be a minimal G-invariant subspace. Now either $\mathrm{rad}(A) = A$ or $\mathrm{rad}(A) = 0$.

If $\mathrm{rad}(A) = A$, either A is totally singular, or $p = 2$, W is orthogonal, and $\dim(A) = 1$ (the set of singular vectors in A is an X- (or G-) invariant subspace, so minimality forces A to be either a 1-space or totally singular). In this exceptional case, we have W even-dimensional and Y of type D_n, since if $p = 2$ and $Y = B_n$ we take W to be the symplectic $2n$-dimensional module. So, since X is simple and $I(A)' = 1$, we have the situation $X \leq B_{n-1} = Y_2 < D_n = Y$, with Y_2 being t-stable (we must be in the case $A = A^t$ here, since otherwise $W = \langle G \cdot A \rangle$ has dimension 2, which would imply $X < A_1$ admits no graph automorphism). As in the proof of Lemma 4.4, there are two possibilities: A. V is irreducible for B_{n-1}; or B. $V|_{B_{n-1}} = V_1 \oplus V_2$. If A holds, then as in Lemma 4.4, the only possibility is U_3 of Table 2 (by induction from the result in [10]). If B holds, then we again get a contradiction as in the proof of 4.4. So this exceptional case gives only U_3.

Suppose A is totally singular and $t \notin Y$ but t acts on W, preserving A. Then $X \leq \mathrm{stab}_Y(A) = P_Y$, a t-stable parabolic subgroup of Y. So $P_Y\langle t\rangle$ acts irreducibly on V. Let $P_Y = Q_Y L_Y$ be the Levi decomposition; then Q_Y has fixed points $V^{Q_Y} \neq 0$ on V, and V^{Q_Y} is an irreducible L_Y-submodule (hence P_Y-submodule) of V by the result of [12]. But t preserves Q_Y, so V^{Q_Y} is in fact a $P_Y\langle t\rangle$- submodule of V, contrary to our statement above that $P_Y\langle t\rangle$ acts irreducibly on V ($V^{Q_Y} \neq V$ since this would imply that L_Y acts irreducibly on V, which is impossible). If A is totally singular, $t \in Y$, and A is t-stable, we have $G \leq \mathrm{stab}_Y(A)$, which is a parabolic subgroup of Y. However, a parabolic subgroup cannot act irreducibly on V.

So if A is totally singular, we must have either $W = A \oplus A^t$ an irreducible G-module, or t not acting on W (in the second case the form on W is trivial). If the form on W is nontrivial, then A is a maximal totally singular subspace (since $\dim(A) = \dim(W)/2$), and $\mathrm{stab}_Y(A) = P_Y$ is a maximal parabolic sugroup such that (with appropriate choice of Borel subgroup, and switching the labels of α_n, α_{n-1} if necessary in the D_n case) $\Pi(Y) - \Pi(L_Y') = \alpha_n$ (where $P_Y = Q_Y L_Y$ is the Levi decomposition). In the case $Y = \mathrm{SL}(W)$, $\mathrm{stab}(A)$ is a parabolic subgroup P_Y of Y such that (with appropriate choice of Borel subgroup) $\Pi(Y) - \Pi(L_Y) = \alpha_j$ for some j. In both cases, $X \leq P_Y$. In the case of a nontrivial form let $j = n$.

Now we want to apply Lemma 4.2; to do so, we need only show that there are only two non-zero $V^i(Q_Y)$, and that they have the same dimension. We know that $X \leq L_Y'$ has only two irreducible summands in V, both of the same dimension. But each quotient $[V, Q_Y^i]/[V, Q_Y^{i+1}]$ is $X < L_Y$-invariant. So in fact only $V^1(Q_Y) = V/[V, Q_Y]$ and $V^2(Q_Y) = [V, Q_Y]/[V, Q_Y, Q_Y]$ can be non-zero; they must have the same dimension because each is isomorphic to one or the other of the V_i, which have the same dimension. So by Lemma 4.2 we only have the possibilities $V \cong W$; $Y = D_4 > X = A_3$ with $\lambda = \lambda_3$ with $V|_X = V_X(\delta_1) \oplus V_X(\delta_3) \cong W|_X$; and $Y = A_n$ for n odd, $L_Y' = A_{n-1}$, $\lambda = \lambda_{(n+1)/2}$. Now for the last case we check Table 1 in [10] and see that there are no simple connected proper subgroups of A_{n-1} (n odd) admitting a graph automorphism which act irreducibly on the A_{n-1}-module with high weight $\lambda_{(n+1)/2}$. So $X = L_Y' = A_{n-1}$. Similarly no simple connected proper subgroup of A_3 admitting a graph automorphism acts irreducibly on the A_3-module with high weight δ_1.

It remains to find t: $t \notin Y$ because otherwise $\dim(A) = \dim(W)/2 = (n+1)/2$, $W = A \oplus A^t$, and X does not stabilize an n-dimensional subspace (but $P_Y > X$ does). Let t' be the standard graph automorphism of $Y = A_n$. Let w_0 be the long word in the Weyl group. Then $t' w_0$ is an outer automorphism of Y which stabilizes the Levi factor of P_Y. So U_1 of Table 2 occurs here.

The automorphism t for the $A_3 < D_4$ case was desribed in the proof of Lemma 4.2. So U_4 of Table 2 occurs here.

This completes the proof of the theorem for the case $\mathrm{rad}(A) = A$.

If $G = X\langle t\rangle$ and $\mathrm{rad}(A) = 0$, then W is orthogonal or symplectic and either $W = A \oplus A^t$, $X \leq I(A)' \times I(A^t)' = Y_1 \times Y_2$, and t stabilizes $Y_1 \times Y_2$, interchanging the two factors; or $A = A^t$, $W = A \perp A^{\perp}$, $X < I(A)' \times I(A^{\perp})' = Y_1 \times Y_2$, and t stabilizes each factor. This is the setup of Lemma 4.4, which tells us we must be in one of the situations listed in the Theorem.

We have now proved Theorem 4.1 under the assumption that $G = X\langle t\rangle$.

4.3. $X = D_4$**.** Assume $X = D_4$ and $G = X\langle s\rangle$ or $X\langle s,t\rangle$. Note that $s \in Y$ because no simple group properly containing D_4 has an outer automorphism of order 3. If $G = X\langle s,t\rangle$ and $V|_X = V_1 \oplus V_2$, then $V_1^s = V_1$, so the X-high weight of V_1 is symmetric with respect to s. But then this high weight is symmetric with respect to t as well, wich means that V is not irreducible as a G-module ($\{(v, v^t) \mid v \in V_1\}$ is a submodule), which is a contradiction. A similar argument shows that whenever G acts irreducibly on W, $W|_X$ has either 3 or 6 irreducible summands. Finally, if $V|_X = V_1 \oplus V_2 \oplus V_3$ with V_i irreducible, then we may assume $G = X\langle s\rangle$. So if $G = X\langle s,t\rangle$, then V has six summands as an X-module.

If G acts on W, we let A be a minimal G-invariant subspace of W if G acts reducibly, and a minimal X-invariant subspace of W if G acts irreducibly on W. If $G = X\langle s,t\rangle$ and t does not act on W, we let A be a minimal X-invariant subgroup if $X\langle s\rangle$ acts irreducibly, and a minimal $X\langle s\rangle$-invariant subspace otherwise.

As before, $\mathrm{rad}(A) = A$ or $\mathrm{rad}(A) = 0$. If $\mathrm{rad}(A) = A$, either A is totally singular, or $p = 2$, W is orthogonal, and $\dim(A) = 1$ (the set of singular vectors in A is an X- (or G-) invariant subspace, so minimality forces A to be a 1-space or totally singular). As before, in this exceptional case we have $\dim(W)$ even and Y of type D_n, and thus t acts on W. So, since X is simple and $I(A)' = 1$, we have the situation $X \leq B_{n-1} = Y_2 < D_n = Y$, with Y_2 being t-stable (we must be in the case $A = \langle G \cdot A \rangle$ here, since otherwise $W = \langle G \cdot A \rangle$ has dimension 3 or 6; but neither A_5 nor A_2 has a subgroup of type $D_4 = X$). We proceed exactly as we did in this case at the beginning of the last section; there are no new examples.

If $G = D_4\langle s \rangle$ and A is totally singular, then, as with $G = X\langle t \rangle$, we have $W = \langle G \cdot A \rangle = A \oplus A^s \oplus A^{s^2}$ irreducible for G (otherwise G is contained in a parabolic subgroup of Y). If the form on W is nontrivial, then we have an X-stable chain $0 < A < A^{\perp} < W$, and $W/A^{\perp} \cong A^*$. Now for any irreducible D_4-module we have $A \cong A^*$, i.e. $A^s \oplus A^{s^2} \cong W/A$ contains a D_4-composition factor isomorphic to A. So $A \cong A^s$ or A^{s^2}, say A^s. But then $A \cong A^s \cong A^{s^2}$, and $W = A \oplus A^s \oplus A^{s^2}$ contains a G-stable proper subspace $\{(a, a^s, a^{s^2}) | a \in A\}$, contrary to the statement above that W is irreducible for G. So in fact the form on W must be trivial; i.e. $Y = A_n$.

A similar argument for $G = D_4\langle s,t \rangle$ shows that W must be irreducible for G and the form on W must be trivial when t acts on W. If t does not act on W, we already know the form must be trivial, but we must consider the case when W is reducible for $X\langle s \rangle$. Assume this is the case; then A is a minimal $X\langle s \rangle$-invariant subspace of W, so $X\langle s \rangle \leq \mathrm{stab}_Y(A) = P_Y$ a parabolic subgroup of Y. But then since P_Y cannot act irreducibly on V and $X\langle s \rangle$ has only two irreducible summands in V, Lemma 4.2 applies. We see that we must have $Y = A_n$, $L'_Y = A_{n-1}$, with $\lambda = \lambda_{(n+1)/2}$ and the restriction of V to L'_Y as in U_1 of Table 2. Let $G_1 = X\langle s \rangle$, $Y_1 = L'_Y$, and $V_1 = V_{Y_1}(\lambda_{(n+1)/2})$. Then we inductively have the situation we are examining in this section: $G_1 < Y_1$, V_1 is an irreducible restricted Y_1-module which is also irreducible for G_1 but not for X, with restricted X-high weights. Since there are no examples for this setup, we have none for $G < Y$.

So $Y = A_n$ and W is irreducible for G (or for $X\langle s \rangle$ when $t \in G$ does not act on W). Consider the action of $X = D_4$ on W. Via the isomorphism $A \cong_X A^*$, X fixes a nondegenerate form on A (this form is orthogonal if $p \neq 2$ by [14, Lemma 79]). Similarly, X fixes a form of the same type on each of its other summands in W, given by the action of s (and t if necessary): $(a^s, b^s) = (a, b)$ for $a, b \in A$. Define a form on W by setting $(a, b) = 0$ for a, b in different summands; then X fixes this form. If the form is orthogonal, choose an orthogonal basis for A; translate this basis by s (and t) to obtain bases of A^s and A^{s^2} (and A^t, A^{ts}, A^{ts^2} if $t \in G$ and t acts on W). The union of these bases is an orthogonal basis for W, and by Witt's Theorem there is an element $\hat{s} \in \mathrm{SO}(W)$ which permutes the X-summands of W as s does. So in fact $X\langle \hat{s} \rangle \cong X\langle s \rangle$ fixes some form, contrary to $Y = A_n$ (we always take Y to be the smallest of $\mathrm{SL}(W), \mathrm{SO}(W), \mathrm{Sp}(W)$ which contains X and whose automorphism group contains G). If the form on A is symplectic (as noted above, this can only happen for $p = 2$), then choose a hyperbolic basis on each X-summand of W and continue as above.

This completes the argument for $\mathrm{rad}(A) = A$.

If $\mathrm{rad}(A) = 0$, $X = D_4$ and $G = X\langle s \rangle$ or $G = X\langle s,t \rangle$, then one of the following holds (notice that t must act on W if it is in G, since $Y \neq A_n$):

A) $A = \langle G \cdot A \rangle$ and $W = A \perp A^{\perp}$, $X \leq I(A) \times I(A^{\perp}) = Y_1 \times Y_2$. As $s \in Y$ and s preserves A and $A^{\perp}$, in fact $X\langle s \rangle \leq I(A) \times I(A^{\perp})$. So either $V|_{Y_1 \times Y_2}$ is irreducible, or $V|_{Y_1 \times Y_2}$ has two summands, interchanged by t (t preserves A and so acts on Y_1 and Y_2).

If $V|_{Y_1 \times Y_2}$ is irreducible, then the possibilities for $V, Y'_1 \times Y'_2$ are as on page 16. In case 1) there, $V = V' \otimes V'' = V_1 \oplus \cdots \oplus V_k$ ($k = 3$ or 6) as X-modules, for some X-modules V' and V'', which must both be reducible as X-modules as on page 16.

If μ_1 is a T_X-high weight of V' and μ_2 a T_X-high weight of V'', then $\mu_1 + \mu_2$ is a T_X-high weight of $V' \otimes V'' = V$. In our case, since s preserves A, it acts on V' and V''. So μ_1^s and $\mu_1^{s^2}$ are also high weights of V'. If μ_1 and μ_2 are not symmetric with respect to s, then it is easy to see that the possible sums $\mu_1^{s^i} + \mu_2^{s^j}$ for $0 \leq i, j \leq 2$ are not all images under $\langle s,t \rangle$ of a single T_X-weight. If both μ_1 and μ_2 are symmetric with respect to s, then so is $\mu_1 + \mu_2$ — but V has no T_X-high weights which are symmetric with respect to s. Finally, if μ_1 is not symmetric with respect to s, and all the T_X-high weights (there are at least two, say μ_2 and μ'_2) of V'' are symmetric, we see that either $\mu_1 + \mu_2$ and $\mu_1 + \mu'_2$ are not conjugate under the action of $\langle s,t \rangle$ (if $\mu_2 \neq \mu'_2$), or there is a T_X-high weight space in V of dimension bigger than 1. Neither is possible.

If $V, Y'_1 \times Y'_2$ are as in case 2) on page 16, then we have $D_4\langle s \rangle < B_n < D_{n+1}$. In fact, though, since t also acts on B_n (if $t \in G$) and B_n has no outer automorphisms, we have $G < B_n$. In [5] it is shown that there are no examples with X acting irreducibly on the natural module for B_n. So X must act reducibly on the natural module for B_n; i.e. we are back in the situation we consider here. So we have no examples, as B_n doesn't appear as an overgroup for $D_4\langle s\rangle$.

If $V|_{Y_1\times Y_2}$ is reducible, then $t\in G$ and $V|_{Y_1\times Y_2}$ has two summands, interchanged by t (t normalizes $Y_1\times Y_2$). Let V' be one of these summands; then, renumbering if necessary, $V'=V_1\oplus V_2\oplus V_3$. So $V_1\oplus V_2\oplus V_3 = V^1\otimes V^2$ for some Y_1-module V^1 and Y_2-module V^2.

These modules V^1 and V^2 must be reducible as X-modules, since otherwise the fact that s acts on them would force the T_X-high weight μ_i of V^i to be s-symmetric, in which case the T_X-high weight $\mu_1+\mu_2$ of V is s-symmetric, which is a contradiction.

Now we argue as above and conclude that one of the V^i, say V^2, must be trivial. But then $(G_0 = X\langle s\rangle, Y_1, V^1)$ must be an example appearing in Table 2; since there are none of this form, we have no examples here.

B) W is irreducible as a G-module and A is an irreducible X-module. Then $W = A\oplus A^s\oplus A^{s^2}$ $(\oplus A^t\oplus A^{ts}\oplus A^{ts^2}$, possibly$) = W_1\oplus\cdots\oplus W_l$ and $X\le I(A)'\times I(A^s)'\times I(A^{s^2})$ $(\times\cdots\times I(A^{ts^2})) = Y_1\times\cdots\times Y_l$. Recall $V =_X V_1\oplus\cdots\oplus V_k$ ($k=3$ for $G=X\langle s\rangle$, 6 for $G=X\langle s,t\rangle$).

We can see by examining Table 1 of [10] that there are no possibilities here for $V|_{Y_1\times\cdots\times Y_l}$ irreducible.

So V is a reducible $Y_1\times\cdots\times Y_l$-module; let V' be a $Y_1\times\cdots\times Y_l$-stable subspace of minimal dimension. Since $X<Y_1\times\cdots\times Y_l$, this implies that $V'=\sum_{i\in I}V_i, I\subseteq\{1,\ldots,k\}, |I|\le k/2$ (since t and s normalize $Y_1\times\cdots\times Y_l$, V'^s is another $Y_1\times\cdots\times Y_l$-stable subspace). If $V'=V_1$ (i.e. $|I|=1$), then we have $V_1 =_X V_1^{(1)}\otimes\cdots\otimes V_1^{(l)}$ for some Y_i-modules $V_1^{(i)}$. But restricted irreducible D_4-modules are not tensor decomposable (2.1), so only one $V_1^{(i)}$ is nontrivial, say $V_1^{(1)}$; i.e. only Y_1 acts nontrivially on V_1. But G permutes the V_i and the Y_i and we see that for every j, V_j is acted on irreducibly by one Y_i, trivially by the others. Then we are in the situation of Lemma 4.3, which tells us $V\cong W$.

If $|I|>1$, we must be in the case $k=6$ (since $|I|\le k/2$). We have (perhaps renumbering the V_i) $V_1\oplus V_2(\oplus V_3) = V^{(1)}\otimes\cdots\otimes V^{(l)}$ for Y_i-modules $V^{(i)}$. If one of the $V^{(i)}$ is reducible as an X-module (via the projection $X\hookrightarrow Y_i$), we conclude as above that only one of the $V^{(i)}$ is nontrivial and apply Lemma 4.3, as otherwise we could conclude that one of the V_j was tensor decomposable as an X-module.

So we assume that each $V^{(i)}$ is irreducible as an X-module, and at least two of them are nontrivial. As s and t permute them, all the Y_i are isomorphic. Note that s and t act on $Y_1\times\cdots\times Y_l$, and again since $X\le Y_1\times\cdots\times Y_l$, V is irreducible for $(Y_1\times\cdots\times Y_l)\langle s,t\rangle$. So if $V'=V_1\oplus V_2$, this implies that $V=V'\oplus V'^s\oplus V'^{s^2}$, with t stabilizing V' and interchanging V_1 and V_2. In this case t must permute the Y_i which act nontrivially. If $V'=V_1\oplus V_2\oplus V_3$, we have $V=V'\oplus V'^t$, with s stabilizing V' and permuting V_1, V_2, and V_3, and permuting the Y_i which act nontrivially on V'.

Note that if $l=6$, since G acts irreducibly on W, all six X-summands of W must be non-isomorphic. In other words, the labelling of the T_X-high weight γ of $W_1=A$ (which is the natural module of Y_1) must be non-symmetric with respect to the action of s and t on the Dynkin diagram for X (if $\gamma=\gamma^\sigma$ for σ the induced action of s or t on the Dynkin diagram, then G preserves a "diagonal" submodule of W). Similarly, if $l=3$ then γ cannot be of the form $a\delta_1+b\delta_2+a\delta_3+a\delta_4$.

Now consider the possibilities for the $V^{(i)}$. The group D_4 appears only once in Table 1 of [10] as a subgroup of a classical group other than A_n (case S$_8$ there), and in that case the restriction of the natural module of Y_i to D_4 has a symmetric high weight. Thus either each of the nontrivial $V^{(i)}$ is the natural module for Y_i, or $X=D_4\cong Y_i$ for every i.

Assume that each of the nontrivial $V^{(i)}$ is the natural module for Y_i. If $V'=V_1\oplus V_2$, we know that t preserves V', permuting the Y_i which act nontrivially. But the high weights of the natural modules of Y_i and Y_i^t are t-conjugate; in other words, their sum is symmetric with respect to t. Since the sum of the high weights for the $V^{(i)}$ is a high weight in $\otimes V^{(i)}$, this gives a t-symmetric high weight in V', which is impossible as t interchanges the two T_X-high weights there. Similarly if $V'=V_1\oplus V_2\oplus V_3$, then s acts on V' and the high weights of the natural modules of Y_1, Y_1^s, and $Y_1^{s^2}$ add up to an s-symmetric high weight in V', which is again impossible.

So the only setup we have not ruled out is $X = D_4 \cong Y_i$ for every i. In this case, the natural module for Y_1 restricts to X as a natural module for D_4, whose high weight does not have six S_3-conjugates. So we know that $l = 3$ with t fixing one of Y_1, Y_2, and Y_3 and interchanging the other two, and $Y = D_{12}$. If $V' = V_1 \oplus V_2$ then t stabilizes V' and so the high weights of the $V^{(i)}$ add up to a t-stable high weight in V', which is impossible as above.

If $V' = V_1 \oplus V_2 \oplus V_3$, then s stabilizes V', we have $V = V' \oplus V'^t$, and s permutes the Y_i which act nontrivially on V' (which implies all three Y_i act nontrivially). But then the T_X-high weights of the $V^{(i)}$ are s-conjugates of each other, and their sum is symmetric with respect to s, which again is a contradiction.

This completes the proof of Theorem 4.1. □

5. THE CASE WHEN t DOES NOT ACT ON W

In this section we consider the cases when X acts irreducibly on W, $t \in G$, and t does not act on W.

Notice that if $G = D_4\langle s,t\rangle$ and D_4 acts irreducibly on W, then the fact that $s \in Y$ forces the D_4-high weight of W to be of the form $a\delta_1 + b\delta_2 + a\delta_3 + a\delta_4$, which implies that t also acts on W. So we need consider only $G = X\langle t\rangle$. The main result is:

Theorem 5.1. *If $G = X\langle t\rangle$, X acts irreducibly on W, and t does not act on W, then we are in situation* U_7, U_8, *or* U_9 *of Table 2, all of which occur.*

Proof. We will use heavily the construction given in Lemma 2.7 of a parabolic subgroup of Y containing a given parabolic subgroup of X (we usually apply it to the Borel subgroup B_X). First we need a Lemma about that embedding; this will be useful in [5] as well:

Lemma 5.2. *If P_Y is a t-stable parabolic subgroup of Y such that $B_X < P_Y$, $U_X < Q_Y$, $T_X < L'_Y$ (where $P_Y = Q_Y L_Y, B_X = U_X T_X$ are Levi decompositions), then one of the simple factors of L'_Y has type A_1; and if this factor corresponds to α_j, then $a_j = 1$. In addition, $a_i = 0$ for $\alpha_i \in \Pi(L'_Y)$, $i \neq j$.*

Proof. With such a setup, 2.10 in [10] implies that $V/[V,Q_Y]$ is an irreducible L'_Y-module.

Since P_Y is t-stable, $[V,Q_Y]$ is $T_X\langle t\rangle$-stable, hence $[V,Q_Y] = [V,U_X]$ ($[V,U_X] \leq [V,Q_Y]$ since $U_X \leq Q_Y$, and we have equality because $V/[V,U_X]$ is an irreducible $T_X\langle t\rangle$-module and $T_X \leq L_Y$). So

$$V/[V,Q_Y] = V/[V,U_X] = V_1/[V_1,U_X] \oplus V_2/[V_2,U_X],$$

which has dimension 2 by Lemma 2.4. So some simple factor of L'_Y has type A_1, the marking on the corresponding node of the Dynkin diagram in the labelling for λ is 1, and all other markings are 0. □

Now embedding a Borel subgroup B_X of X in a parabolic subgroup of Y via the U_X-level construction of Lemma 2.7 gives a parabolic subgroup P_Y which by Lemma 2.8 is t-stable. Let $P_Y = Q_Y L_Y$ be the Levi decomposition produced in the construction of Lemma 2.7.

The group Y is a simple algebraic group of classical type, and either $t \in Y$ or t is an outer automorphism of Y. If Y is of type B_n or C_n, then Y has no outer automorphisms, so $t \in Y$ and t acts on W. If $Y = D_n$ and t is outer, then t acts on the natural module W (which has high weight λ_1). Since we are assuming t does not act on W, we must therefore have $Y = A_n$.

Since t acts on V, we have a t-symmetric T_Y-high weight for V, which imposes strong restrictions as $Y = A_n$.

By Lemma 5.2, one of the factors L_1 of L'_Y must be of type A_1, and if this factor corresponds to $\alpha_j \in \Pi(Y)$, then $a_i = \delta_{ij}$ for every i such that $\alpha_i \in \Pi(L'_Y)$. The fact that P_Y is t-stable implies that if α_j corresponds to an A_1-factor of L'_Y, then so does α_{n-j+1}. Since λ is also t-stable, $a_j = a_{n-j+1}$. But this says that either there are two non-zero labels on $\Pi(L'_Y)$ (which is impossible by Lemma 5.2), or $j = n-j+1$. So we must in fact have an L_1-factor of L'_Y corresponding to $\alpha_{(n+1)/2}$ (in particular, n must be odd).

We noted in the proof of Lemma 2.8 that the dimension of U_X-level i of W is the same as the dimension of U_X-level $l_\delta - i$, where l_δ is the level of the low weight (l_δ is also minimal with respect to $[W, U_X^{l_\delta+1}] = 0$). This implies that the U_X-level corresponding to the A_1-factor with $\Pi(L_1) = \{\alpha_{(n+1)/2}\}$ must be level $l_\delta/2$. So to prove that a given high weight δ for $W|_X$ gives no examples, it suffices to show that $W_{l_\delta/2}$ has dimension different than 2. In particular, if l_δ is odd, we have nothing to check, since then level $l_\delta/2$ does not exist.

We rely heavily on the results in [15] (for $X = A_m$) and [9] (for $X = D_m$ and $X = E_6$) that all weights which appear in the Weyl module for X with high weight δ also appear in $W = V_X(\delta)$. We use induction on the height of δ in the weight lattice, based on the fact that if μ is a weight in $V_X(\delta)$ (i.e. $V_X(\delta)_\mu \neq 0$) at level l, and ν is a weight in $V_X(\mu)$ at level k, then ν is a weight in $V_X(\delta)$ at level $l+k$. Using this fact, the inductive step is trivial.

5.1. $X = A_m$**.** Assume X is of type A_m. Let $\delta = d_1\delta_1 + d_2\delta_2 + \cdots + d_m\delta_m$ ($d_i \geq 0$) be the T_X-high weight of W. Since t does not act on W, $d_i \neq d_{m-i+1}$ for some i. First we will establish that δ has a fundamental dominant weight or 0 as a subdominant weight; following that we will show that A_m-modules with fundamental dominant weights have at least 3 weights at the "middle" level except in a few cases; finally, we will deal with these few cases and with the case $\delta \succ 0$.

Lemma 5.3. *Either $\delta \succ \delta_i$ for some i, $1 \leq i \leq m$, or $\delta \succ 0$.*

Proof. See exercise III.13 in [7]. ☐

So if we show that $V_X(\delta_i)$ has three or more weights at level $l_{\delta_i}/2$, then all δ with δ_i as a subdominant weight will be ruled out. The next step is:

Lemma 5.4. *If δ is a non-zero dominant weight for X which is not symmetric with respect to the action of t, such that level $l_\delta/2$ has exactly two non-zero weights, then we are in one of the following situations (perhaps after reversing the labelling):*

1. *$X = A_2$, with $\delta = 3\delta_1$, $\delta = 2\delta_2$, $\delta = 2\delta_1 + \delta_2$, or $\delta = 3\delta_1 + \delta_2$;*
2. *$X = A_3$, with $\delta = 2\delta_1$; or*
3. *$X = A_4$, with $\delta = \delta_2$.*

Proof. The proof is long and somewhat tedious; we give a sketch here. By the last lemma, δ has some δ_i or 0 as a subdominant weight. We may assume $i \leq (m+1)/2$ by reversing the labelling if necessary.

If we restrict the usual ordering on the weights of T_X ($\mu \succ \nu$ if $\mu - \nu$ is a sum of positive roots) to the dominant weights, then some of the weights in question have the nice property that they have unique immediate successors; e.g. $\delta_1 \prec \delta_2 + \delta_m$; $0 \prec \delta_1 + \delta_m \prec \delta_2 + \delta_{m-1}$ ($m \geq 3$), or $\delta_1 \prec 2\delta_2 \prec 2\delta_1 + \delta_2$ ($m = 2$). We use this fact, the last lemma, and the fact that δ is not t-symmetric, to exhibit at least 3 weights (or show that there is at most 1 weight) at the middle level when δ is not one of the weights listed in the lemma. ☐

So we need consider only those δ listed in the statement of the Lemma, as the middle level must be of dimension 2, and the only δ with a single weight at the middle level have a weight space of dimension 1 for that weight.

(1) Assume $X = A_2$ and $\delta = 3\delta_1$. Then Y is of type A_9, and if $P_Y = Q_Y L_Y$ is the parabolic subgroup containing the Borel subgroup B_X of X, given as the stabilizer of the U_X-levels of W, then P_Y corresponds to $\{\alpha_3, \alpha_5, \alpha_7\} \subseteq \Pi(Y)$. For $\lambda = \sum a_i\lambda_i$ the T_Y-high weight of V, by Lemma 5.2 and the comments which follow it, we have $a_5 = 1$ and $a_3 = a_7 = 0$. Now by Lemma 2.9, we have $[V, Q_y] = [V, Q_X]$ and, since $Q_x \leq Q_Y$, we have $[V, Q_X, Q_X] \leq [V, Q_Y, Q_Y]$. Thus $\dim([V, Q_Y]/[V, Q_Y, Q_Y]) = \dim(V^2(Q_Y)) \leq \dim(V^2(Q_X))$. Then applying Lemma 2.4 to Q_X, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 2\dim(V^1(Q_X)) = 2\dim(V^1(Q_Y)) = 4$. But then $a_i > 0$ for some $i \neq 5$ violates this bound. So in fact $V = \bigwedge^5(W)$, of dimension $\binom{10}{5} = 252$.

A high weight vector of $\bigwedge^5(W|_X)$ is $w_1 \wedge w_2 \wedge w_3 \wedge w_4 \wedge w_5$, where $w_1 \in W_\delta$, $w_2 \in W_{\delta-\beta_1}$, $w_3 \in W_{\delta-2\beta_1}$, $w_4 \in W_{\delta-\beta_1-\beta_2}$, and $w_5 \in W_{\delta-2\beta_1-\beta_2}$. So a T_X-high weight of V is $5\delta - 6\beta_1 - 2\beta_2 = 5\delta_1 + 2\delta_2$. The dimension of the Weyl module with this high weight is 81, so $\dim(V_1) \leq 81 < 126 = \frac{1}{2}\dim(V)$, which is a contradiction. So $\delta = 3\delta_1$ gives no examples.

If $\delta = 2\delta_2$, then Y is of type A_5 and $P_Y = Q_Y L_Y$ as above corresponds to $\alpha_3 \in \Pi(Y)$. As above, any non-zero a_i other than $a_3 = 1$ contradicts $\dim(V^2(Q_Y)) \leq 4$ (since λ is symmetric with respect to the action of t). So $V = \bigwedge^3(W)$, of dimension 20. As above, we can compute the weight of a T_X-high weight vector of V; we find that it is $3\delta_1$ or $3\delta_2$. The A_2-modules with these high weights each have dimension $10 = \frac{1}{2}\dim(V)$ if $p \neq 2, 3$. So here we have case U$_7$ of Table 2: $p \neq 2, 3$, $X = A_2$, $Y = A_5$, $V_1 = V_{A_2}(3\delta_1)$, and $V = V_{A_5}(\lambda_3)$.

If $\delta = 2\delta_1 + \delta_2$, then although there are only 2 weights at the middle level ($\delta - \beta_1 - 2\beta_2$ and $\delta - 2\beta_1 - \beta_2$), one of them ($\delta - 2\beta_1 - \beta_2$) has a weight space of dimension 2 in all characteristics. So $\dim(W_3) = 3$, and there is no A_1-factor of L'_Y corresponding to the "middle" level.

Finally, if $\delta = 3\delta_1 + \delta_2$, then as above the weights in the middle level have weight spaces of dimension 2 or more except in characteristic 5. So for $p \neq 5$ there are no examples here. If $p = 5$, then Y has type A_{17}, and $a_9 = 1$ is the only non-zero a_i for $\alpha_i \in \Pi(L'_Y)$. Now if some $a_i \neq 0$ for $\alpha_i \in \Pi(Y) - \Pi(L'_Y)$, we obtain a contradiction to $\dim(V^2(Q_Y)) \leq 4$. So in fact $V = \bigwedge^9(W)$, of dimension 48,620. But no restricted A_2-module has dimension 24,310 (Weyl's character formula shows that the dimension of the Weyl module with a restricted high weight is at most the dimension of the Weyl module with high weight $(p-1)\rho$, and this dimension is 125). So there are no examples here.

(2) Suppose $X = A_3$ and $\delta = 2\delta_1$. Then Y is of type A_9 and $a_5 = 1$. By Lemmas 2.4 and 2.9 as before, $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 3\dim(V^1(Q_X)) = 3\dim(V^1(Q_Y)) = 6$, when $P_Y = Q_Y L_Y$ is the parabolic subgroup of Y containing B_X, constructed as the stabilizer of the U_X-levels of W. In this situation P_Y corresponds to $\{\alpha_3, \alpha_5, \alpha_7\} \subseteq \Pi(Y)$. Then $\dim(V^2(Q_Y)) \leq 6$ implies $a_i = 0$ for $i \neq 5$, as otherwise too many non-zero weight spaces appear in $V^2(Q_Y)$. So $V = \bigwedge^5(W)$, of dimension 252; as above, we can compute a T_X-high weight of V and obtain $2\delta_1 + 2\delta_2$ or $2\delta_2 + 2\delta_3$. The A_3-modules with these high weights have dimension $126 = \frac{1}{2}\dim(V)$ for $p \neq 2, 5$, so here we have case U$_8$.

(3) Let $X = A_4$ and $\delta = \delta_2$. Then $Y = A_9$ and $a_5 = 1$. By examining the stabilizer of the U_X-levels as above, we can conclude that $a_i = 0$ for $i \neq 1, 5, 9$. So the possible high weight is $a\lambda_1 + \lambda_5 + a\lambda_9$. Then if we let P_X be the parabolic subgroup corresponding to $\{\beta_1, \beta_4\}$, and embed P_X in a parabolic subgroup P_Y as usual, we find that $a \neq 0$ forces a too-large $V^2(Q_Y)$. So again, $V = \bigwedge^5(W)$, of dimension 252. Again we compute a high weight of $\bigwedge^5(W|_X)$, obtaining $\delta_2 + 2\delta_4$ or $2\delta_1 + \delta_3$. The A_4-modules with these high weights have dimension 126 except in characteristics 2 and 5. So here we have the last example given in Table 2.

This completes the examination of the cases for $X = A_m$.

5.2. $X = D_m$**.** We must establish a result analogous to the first lemma of the last subsection, narrowing the range of possibilities for δ we must examine.

Lemma 5.5. *If $\delta \neq \delta_1$ is a non-zero dominant weight of T_X, then δ has one of δ_2, δ_3, δ_{m-1}, or δ_m (or $\delta_3 + \delta_4$ if $X = D_4$) as a subdominant weight.*

Proof. By exercise III.13 in [7], δ has one of 0, δ_1, δ_{m-1}, or δ_m as a subdominant weight.

The unique successor to 0 in the partial order on the dominant weights is δ_2, so if $\delta \succ 0$, then $\delta \succ \delta_2$ (since $\delta \neq 0$).

Assume $\delta \succ \delta_1$. We assumed $\delta \neq \delta_1$, and adding positive roots to get a dominant weight forces $\delta \succ \delta_3 \succ \delta_1$ if $m > 4$, or $\delta \succ \delta_3 + \delta_4$ if $m = 4$. So in fact $\delta \succ \delta_3$. ☐

Note that $\delta = \delta_1$ and $\delta = \delta_2$ are not possibilities, since δ is not symmetric with respect to t (as t does not act on W). The two immediate successors to δ_2 in the partial order on the dominant weights are $2\delta_1$ and δ_4 (or, if $m = 4$, $2\delta_3$ or $2\delta_4$; if $m = 5$, $\delta_4 + \delta_5$). So if $m > 5$ and $\delta \succ \delta_2$, then $\delta \succ 2\delta_1$ or $\delta \succ \delta_4$. If we can show that $V_{D_4}(\delta_3 + \delta_4)$, $V_{D_4}(2\delta_3)$, $V_{D_5}(\delta_4 + \delta_5)$, $V_{D_m}(2\delta_1)$ and $V_{D_m}(\delta_i)$ for $i \in \{3, 4, m-1, m\}$ have at least three weights at their middle levels, we will be done.

The first possibilities listed ($2\delta_3$ and $\delta_3 + \delta_4$ for $m = 4$, $\delta_4 + \delta_5$ for $m = 5$, $2\delta_1$, δ_3, and δ_4) are relatively easily dealt with by simply listing three weights at the middle level. For example, in $V_X(\delta_3)$ the low weight is at level $6m - 12$, so the middle level is level $3m - 6$. At this level, for $m \geq 6$ we have the weights $\pm(\delta_{m-1} - \delta_m)$ (by first subtracting $\beta_3 + \cdots + \beta_{m-1}$, then $\beta_2 + \cdots + \beta_{m-2} + \beta_m$, and finally $\beta_1 + \cdots + \beta_{m-1}$ or $\beta_1 + \cdots + \beta_{m-2} + \beta_m$), and

$$\delta_3 - 2\beta_1 - 2\beta_2 - 3\beta_3 - 3\beta_4 - \cdots - 3\beta_{m-2} - \beta_{m-1} - \beta_m = -2\delta_1 + \delta_2 - \delta_{m-2} + \delta_{m-1} + \delta_m.$$

The cases for $m \leq 5$ and the other weights are similar.

Listing three weights at the middle level for $\delta = \delta_m$ is more difficult ($\delta = \delta_{m-1}$ is the same argument by symmetry). The low weight here is at level $m(m-1)/2$, so we must check level $m(m-1)/4$. The level only exists if $m = 4k$ or $m = 4k+1$ for some k. For $m = 8$ or 9, we can check directly, writing down at least 3 weights at this middle level. So assume $m \geq 12$. Then

$$\begin{aligned} \delta' &= \delta_m - (\beta_2 + \cdots + \beta_{m-2} + \beta_m) - (\beta_3 + \cdots + \beta_{m-1}) \\ &= \delta_1 - \delta_3 + \delta_m \end{aligned}$$

is a weight of $V_X(\delta_m)$ at level $2m-5$. Let $P_X \leq X$ be the obvious parabolic subgroup with Levi factor L such that L' is of type D_{m-4}, and let $\{\gamma_1, \dots, \gamma_{m-4}\}$ be the fundamental dominant weights of L'. Then $\delta'|_{T_{L'}} = \delta_m|_{T_{L'}} = \gamma_{m-4}$, and by induction this D_{m-4}-weight has at least three weights at level $(m-5)(m-4)/4$. This implies δ' has at least three weights at this level, which in turn implies that δ_m has at least three at level $(2m-5)+(m-5)(m-4)/4 = m(m-1)/4$, which is its middle level. So if $\delta \succ \delta_m$ for $m \geq 6$, then δ has at least 3 weights at its middle level.

For $m = 4$ or 5, $V_X(\delta_m)$ has only two weights, both with weight spaces of dimension 1, at level $m(m-1)/4$. Assume $m = 4$. Then Y has type A_7, and when we embed the Borel subgroup B_X in a parabolic subgroup $P_Y = Q_Y L_Y$ of Y, the subgroup we obtain corresponds to $\alpha_4 \in \Pi(Y)$. So $a_4 = 1$. By Lemmas 2.4 and 2.9, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 4\dim(V^1(Q_X)) = 4\dim(V^1(Q_Y)) = 8$. Since t acts on V, we have $a_1 = a_7$, $a_2 = a_6$, and $a_3 = a_5$. Now if more than one of a_1, a_2, and a_3 is non-zero, we obtain a contradiction to $\dim(V^2(Q_Y)) \leq 8$ by Lemma 2.4. So at most one of these is non-zero. Let the T_X-high weight of V_1 be $b_1\delta_1 + b_2\delta_2 + b_3\delta_3 + b_4\delta_4$.

Next consider the embedding of the (t-stable) parabolic subgroup $P_X = Q_X L_X$ of X corresponding to $\{\beta_1\} \subseteq \Pi(X)$ in a parabolic subgroup $P_Y = Q_Y L_Y$ of Y via Q_X-levels. Here P_Y corresponds to $\{\alpha_3, \alpha_5\} \subseteq \Pi(Y)$. Since by Lemma 2.9 we have $V/[V, Q_Y] = V_1/[V_1, Q_X] \oplus V_2/[V_2, Q_X]$, we can compute dimensions (since an irreducible restricted A_1-module has the same dimension as the Weyl module with the same high weight) and obtain $2(b_1+1) = (a_3+1)(a_5+1) = (a_3+1)^2$, or $b_1 = \frac{(a_3+1)^2}{2} - 1$. But b_1 is integral, so this implies $a_3 \neq 0$ (indeed, a_3 must be odd). So $a_1 = a_2 = 0$ by our above comment that only one of these three can be non-zero.

Our penultimate t-stable parabolic subgroup to consider in this case is the one corresponding to $\{\beta_2\} \subseteq \Pi(X)$. When we embed this via Q_X-levels, we have $P_Y = Q_Y L_Y$ corresponding to $\{\alpha_2, \alpha_4, \alpha_6\} \subseteq \Pi(Y)$. Here again, by Lemmas 2.4 and 2.9, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 6\dim(V^1(Q_X)) = 6\dim(V^1(Q_Y)) = 12$ (we have $\dim(V^1(Q_Y)) = 2$ because we discovered above that $a_2 = a_6 = 0$). Since $a_3 \neq 0$, we have the high weights $\lambda - \alpha_3$ and $\lambda - \alpha_5$ in $V^2(Q_Y)$, each giving L'_Y-modules of dimension 6 (unless $p = 2$, in which case each has dimension 4). If $a_3 \neq p-2$, then $\lambda - \alpha_3 - \alpha_4$ has a 2-dimensional weight space in V, which implies that $\lambda - \alpha_3 - \alpha_4$ is another high weight in $V^2(Q_Y)$. But this contradicts $\dim(V^2(Q_Y)) \leq 12$, if $p \neq 2$. So $a_3 = p-2$, or $p = 2$. Assume $a_3 = p-2$. Above we had $b_1 = \frac{(a_3+1)^2}{2} - 1$. Since $b_1 \leq p-1$, this gives $\frac{(p-1)^2}{2} - 1 \leq p-1$, or $p \leq 4$. So if $0 \neq a_3 = p-2$, we have $p = 3$ and $a_3 = 1$; otherwise $p = 2$ and $a_3 = 1$. The $p = 2$ case cannot occur as the A_7-module with this high weight has dimension 64 512; the largest restricted irreducible D_4-module has dimension 4 096.

Now for one last t-stable parabolic subgroup of X: Let P_X correspond to the subset $\{\beta_2, \beta_3, \beta_4\}$ of $\Pi(X)$. When we embed this in a parabolic subgroup P_Y of Y via Q_X-levels, we have P_Y corresponding to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7\} \subseteq \Pi(Y)$. By Lemmas 2.4 and 2.9 again, we have $\dim(V^2(Q_Y)) \leq \dim(V^2(Q_X)) \leq 6\dim(V^1(Q_X)) = 6\dim(V^1(Q_Y)) = 6 \cdot 4 \cdot 4 = 96$. But $a_4 \neq 0$, so $\lambda - \alpha_4$ is a high weight in $V^2(Q_Y)$, giving an L'_Y-module of dimension $\binom{5}{2}^2 = 100$. This is a contradiction, so we get no examples in this $X = D_4$, $\delta = \delta_4$ case.

If $m = 4$ and $\delta \succ \delta_4$, $\delta \neq \delta_4$, then $\delta \succ \delta_1 + \delta_3$, which has at its middle level (level 6) more than three weights ($\delta - 2\beta_1 - 2\beta_2 - 2\beta_3$, $\delta - \beta_1 - 2\beta_2 - 2\beta_3 - \beta_4$, and $\delta - \beta_1 - 2\beta_2 - \beta_3 - 2\beta_4$, for example). So we in fact get no examples for $m = 4$, $\delta \succ \delta_4$.

For $m = 5$ and $\delta = \delta_5$, we have Y of type A_{15}, and by embedding B_X into a parabolic subgroup $P_Y = Q_Y L_Y$ of Y via Q_X-levels, we obtain the subgroup corresponding to $\{\alpha_4, \alpha_6, \alpha_8, \alpha_{10}, \alpha_{12}\} \subseteq \Pi(Y)$. Here $\dim(V^2(Q_Y)) \leq 5\dim(V^1(Q_Y)) = 10$ by Lemmas 2.4 and 2.9, which implies $a_3 = a_5 = a_7 = a_1 a_2 = 0$. We know $a_4 = a_6 = 0$ and the a_i are symmetric ($a_i = a_{16-i}$).

Now we embed the parabolic subgroup of X corresponding to $\{\beta_3,\beta_4,\beta_5\} \subseteq \Pi(X)$. Then $P_Y = Q_Y L_Y$ corresponds to $\Pi(Y) - \{\alpha_4,\alpha_8,\alpha_{12}\}$, so L_Y' is a product of four groups of type A_3. Since a_1, a_2, a_8, a_{14}, and a_{15} are the only possible non-zero coefficients of λ, $\dim(V/[V,Q_Y]) = (\dim(V_{A_3}(a_1\gamma_1 + a_2\gamma_2)))^2$ (where the γ_i are the fundamental dominant weights of A_3). By Lemmas 2.4 and 2.9, we have $\dim(V^2(Q_Y)) \leq 6\dim(V^1(Q_Y))$, but $\lambda - \alpha_8$ is a high weight in $V^2(Q_Y)$, giving an L_Y'-module of dimension $4\cdot 4\cdot \dim(V^1(Q_Y))$, which is a contradiction. So we have no examples for $m = 5$, $\delta = \delta_5$.

If $m = 5$ and $\delta \succ \delta_5$, $\delta \neq \delta_5$, then $\delta \succ \delta_1 + \delta_4$, which has more than three weights at its middle level as in the $m = 4$ case above. So there are no examples for $m = 5$, $\delta \succ \delta_5$.

This completes the argument for $X = D_m$.

5.3. $X = E_6$**.** Assume $X = E_6$. We again need a few "small" weights, among which every dominant E_6-weight has a subdominant weight.

Lemma 5.6. *If* $\delta = d_1\delta_1 + d_2\delta_2 + d_3\delta_3 + d_4\delta_4 + d_5\delta_5 + d_6\delta_6$ *is a non-zero dominant weight of* T_X*, then* δ *has one of* δ_1, δ_6*, or 0 as a subdominant weight.*

Proof. See problem III.13 in [7]. ☐

Assume $\delta \succ 0$. We know $\delta \neq 0$ as X acts irreducibly on W; adding positive roots to 0, the unique lowest dominant weight in the order is δ_2. Continuing to add roots, the unique successor to δ_2 in the partial order on dominant weights is $\delta_1 + \delta_6$. As $\delta \neq \delta_2$ (since t does not act on W), we have $\delta \succ \delta_1 + \delta_6$. The weight $\delta_1 + \delta_6$ has 0 as a weight at level 16; a simple check show that there are at least two other weights at this level. So if $\delta \succ 0$ we have no examples.

If $\delta \succ \delta_1$ ($\delta \succ \delta_6$ is the same argument by symmetry), then δ has at least three weights at its middle level because δ_1 does: The middle level for δ_1 is 8 (as the low weight is at level 16), and at level 8 are the three weights

$$\delta_1 - \beta_1 - \beta_2 - 2\beta_3 - 2\beta_4 - \beta_5 - \beta_6 = \delta_1 - \delta_3 + \delta_5 - \delta_6,$$
$$\delta_1 - \beta_1 - \beta_2 - \beta_3 - 2\beta_4 - 2\beta_5 - \beta_6 = \delta_3 - \delta_5, \text{ and}$$
$$\delta - 2\beta_1 - \beta_2 - 2\beta_3 - 2\beta_4 - \beta_5 = -\delta_1 + \delta_6.$$

So there are no examples for $X = E_6$.

This completes the proof of Theorem 8.1. ☐

What remains to be considered to complete the proof of Theorem 1 are the cases in which X acts irreducibly on W, and W is t-stable (if $t \in G$). This will be completed in part II ([5]).

TABLE 1. Examples arising from the connected case

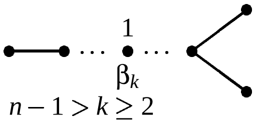
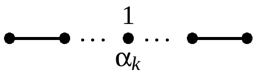
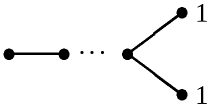
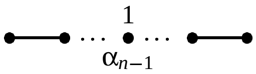
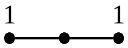
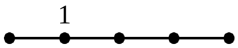
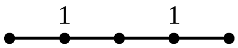
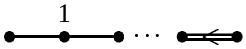
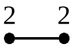
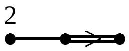
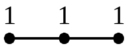
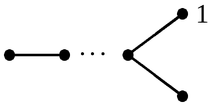
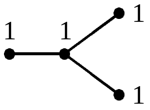
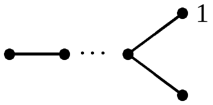
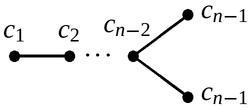
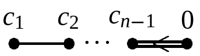
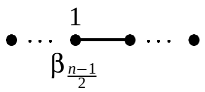
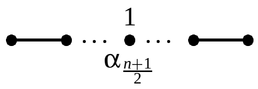
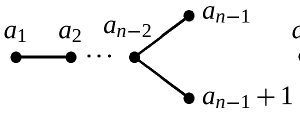
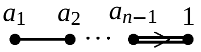
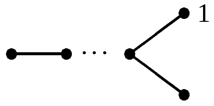
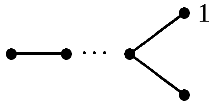
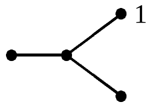
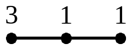
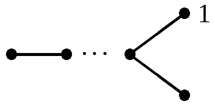
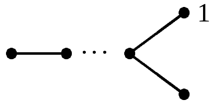
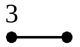
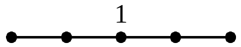
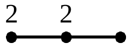
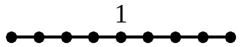
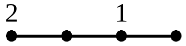
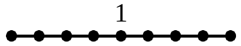
No.	X	Y	$W\|_X$	$V\|_X$	$V\|_Y$	char(K)
I_4	D_n	A_{2n-1} $(n \geq 4)$	δ_1	 $n-1>k\geq 2$		$p \neq 2$
I_5	D_n	A_{2n-1} $(n \geq 4)$	δ_1			$p \neq 2$
I_6	A_3	A_5	δ_2			$p \neq 2$
II_1	A_5	C_{10}	δ_3			$p \neq 2$
S_1	A_2	B_3	$\delta_1+\delta_2$			$p=3$
S_7	A_3	D_7	$\delta_1+\delta_3$			$p=2$
S_8	D_4	D_{13} $(s,t \in Y)$	δ_2			$p=2$
MR_4	D_n	C_n	δ_1			$p=2$

TABLE 2. New Examples

No.	X	Y	$W\|_X$	$V_1\|_X$	$V\|_Y$	char(K)
U_1	A_{n-1} (n odd)	A_n	usual			any
U_2	D_n	B_n	usual			below
				with $a_i + a_j \equiv i - j \pmod p$ whenever a_i and a_j are non-zero coefficients with only 0's between them and $i < j < n$; and $2a_i \equiv -2(n-i) - 1 \pmod p$ for a_i the last non-zero coefficient before $a_n = 1$.		
U_3	D_n	D_{n+1}	usual			any
U_4	A_3	D_4	$\delta_1 \oplus \delta_3$			any
U_5	A_3	D_{10}	$2\delta_2$			$p \neq 2, 3, 5, 7$
U_6	D_m ($m > 3$)	C_m, B_m	δ_1			$p = 2$
U_7	A_2	A_5	$2\delta_1$			$p \neq 2, 3$
U_8	A_3	A_9	$2\delta_1$			$p \neq 2, 5$
U_9	A_4	A_9	δ_2			$p \neq 2, 5$

REFERENCES

1. Roger W. Carter, *Simple groups of Lie type*, Wiley, London/New York/Sydney/Toronto, 1972.
2. Eugene B. Dynkin, *Some properties of the weight system of a linear representation of a semisimple Lie group*, Dokl. Akad. Nauk SSSR **71** (1950), no. 2, 221–224, in Russian.
3. ______, *Maximal subgroups of the classical groups*, Amer. Math. Soc. Transl. Ser. 2 **6** (1957), 245–378.
4. ______, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. Ser. 2 **6** (1957), 111–244.
5. Ben Ford, *Overgroups of irreducible linear groups, II*, (to appear).
6. ______, *Irreducible restrictions of representations of the symmetric groups*, Bull. London Math. Soc. (1995), (to appear).
7. James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer, New York/Heidelberg/Berlin, 1972.
8. George J. McNinch, *Complete reducibility of small modules*, Ph.D. thesis, University of Oregon, (to appear).
9. Alexander A. Premet, *Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic*, Math. USSR-Sb. **61** (1988), 167–183.

10. Gary M. Seitz, *The maximal subgroups of classical algebraic groups*, Mem. Amer. Math. Soc. **67** (1987), no. 365, 1–286.
11. ______, *Maximal subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **90** (1991), no. 441, 1–197.
12. Stephen D. Smith, *Irreducible modules and parabolic subgroups*, J. Algebra **75** (1982), 286–289.
13. Robert Steinberg, *Representations of algebraic groups*, Nagoya Math. J. **22** (1963), 33–56.
14. ______, *Lectures on Chevalley groups*, Yale University, 1967.
15. Irene D. Suprunenko, *The invariance of the weight system of irreducible representations of algebraic groups and Lie algebras of type* A_l *with restricted highest weights, under reduction modulo* p, Vestsī Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk **2** (1983), 18–22 (Russian).
16. Donna M. Testerman, *Irreducible subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **75** (1989), no. 390, 1–190.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, BOX 354350, SEATTLE, WA 98195-4350
E-mail address: `ford@math.washington.edu`