Self-Conjugate t-core Partitions, Sums of Squares, and p-blocks of A_n

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Abstract

We prove that if t is an integer with $t = 8$ or $t \geq 10$, then every integer $n > 2$ has a self-conjugate t-core partition. This result has consequences in the representation theory of alternating groups, and has a version as a theorem about the representation of integers by sums of squares. We also give an infinite sequence of integers that have no self-conjugate 9-core partition.

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1 Introduction

The k-tuple $\lambda = (l_1, l_2, \ldots, l_m)$ of positive integers l_i is a *partition* of the natural number n if the sum of the l_i is n, and the l_i are non-increasing.

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We often collect like parts together and write $\lambda = (\lambda_1^{a_1}, \lambda_2^{a_2}, \ldots, \lambda_k^{a_k})$, with $\lambda_i > \lambda_{i+1} > 0$.

Partitions arise naturally in many contexts, for instance as indexing sets (of irreducible representations, p-blocks, etc.) in the representation theory of certain groups.

We use the symbol $[\lambda]$ to represent the *Ferrers-Young diagram* of the partition λ; $[\lambda]$ consists of n nodes $\bullet$ placed in decreasing rows, as illustrated by the following example. Let $\lambda = (5^2, 2^2)$ (here $n = 14$); then the Young diagram is:

$$[\lambda] = \begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & & \\ \bullet & \bullet & & & \end{array}$$

By the *rim* of $[\lambda]$ we shall mean the collection of nodes which form the "south-east" border of the diagram; that is, a node n of $[\lambda]$ is a member of the rim if there is no node of $[\lambda]$ which is in the row below n and in the column to the right of n. The diagram above has eight nodes in its rim.

The modular (characteristic p) irreducible representations D^λ of the symmetric group S_n are indexed by the *p-regular* partitions λ of n — those with $a_i < p$ for every i.

The partition λ is a t-core (for a natural number t) if it is impossible to remove t consecutive nodes from the rim of $[\lambda]$ and leave a diagram for a partition. For example, the partition $(5^2, 2^2)$ pictured above is a 5-core but not a 3-core. For primes p, the p-core partitions are important in the modular representation theory because the p-blocks of S_n are indexed by the p-cores of $n - kp$, for $k = 0, 1, \ldots$. Similar applications in the quantum settings are of interest for all t.

In [1], Granville and Ono proved the existence of t-core partitions for every integer n, for $t \geq 4$. This established the existence of defect-zero p-blocks for the symmetric groups S_n and thus for the alternating groups A_n.

If p is an odd prime, a simple application of Clifford's Theory tells us that D^λ restricts irreducibly to $A_n < S_n$ if $D^\lambda \otimes \mathrm{sgn}_n \not\cong D^\lambda$, and restricts as the sum of two non-isomorphic irreducible modules if $D^\lambda \otimes \mathrm{sgn}_n \cong D^\lambda$. The "Mullineux Conjecture" (see proofs in [2–4]) tells us that $D^\lambda \otimes \mathrm{sgn}_n \cong D^\lambda$ precisely when λ is fixed under a bijection on the set of p-regular partitions called the "Mullineux map": $m(\lambda) = \lambda$. Andrews and Olsson studied Mullineux fixed points in [5], in their search for evidence for the Mullineux conjecture. (Benson determined the characteristic-2 D^λ which split on restriction to A_n in [6]. In this case, sgn_n is the trivial representation, so the setting is quite different.)

From [7, Prop. 12.2], we know that every p-block of S_n restricts to A_n as a single p-block, except perhaps for those of defect 0 (that is, the irreducible blocks — blocks corresponding to p-core partitions of n); and of these, we know by the previous paragraph that those partitions λ which are Mullineux-fixed are precisely those whose corresponding irreducible representations split on restriction to A_n. This explains one motivation for the question of the existence of (Mullineux-) self-conjugate p-cores.

We can now state the main results:

Theorem 1 *Let n be an integer greater than 2, and let t be an integer with $t = 8$ or $t \geq 10$. Then there exists a Mullineux self-conjugate t-core partition of n.*

Corollary 1.1 *If $n > 2$ and $p > 7$ is prime, then S_n has a defect 0 p-block which restricts to A_n as the sum of two defect 0 p-blocks.*

Corollary 1.2 *Let t be an odd positive integer. If $k \geq \frac{t^2-1}{24} + 3$, then there exist integers $a_1, a_2, \ldots, a_{(t-1)/2}$ such that*

$$kt = \sum_{i=1}^{i=\frac{t-1}{2}} a_i^2,$$

with $a_i \equiv i \pmod{t}$ for each i.

Note: Since

$$\sum_{i=1}^{i=\frac{t-1}{2}} i^2 = \frac{t^2-1}{24}t$$

and

$$\sum_{i=1}^{i=\frac{t-3}{2}} i^2 + (-t + \frac{t-1}{2})^2 = (\frac{t^2-1}{24} + 1)t,$$

this last Corollary establishes that every multiple of t that might conceivably be represented by a sum of squares of integers of the appropriate residues modulo t *is* so represented, except for $(\frac{t^2-1}{24} + 2)t$.

PROOF. Corollary 1.2 follows from Theorem 1 simply by noting that equation (2) below can be rewritten (by completing the square) as

$$(\frac{t^2-1}{24} + n)t = \sum_{i=1}^{(t-1)/2} (tx_i + i)^2.$$

□

Mullineux himself proved that the bijection he defined and ordinary conjugation coincide on p-cores [8]; this is true for non-prime t as well (since for a t-core, the t-rim and ordinary rim coincide). Thus Theorem 1 will be proved by demonstrating the existence of self-conjugate t-cores for the stated ranges of n and t. Finally, Garvan, Kim, and Stanton in [9, Equation 7.4] show that the number of self-conjugate t-core partitions of n is equal to the number of solutions in integers x_i to

$$n = \sum_{i=1}^{\frac{t}{2}} tx_i^2 + (2i-1)x_i \qquad \text{for even } t \tag{1}$$

$$n = \sum_{i=1}^{\frac{t-1}{2}} tx_i^2 + 2ix_i \qquad \text{for odd } t \tag{2}$$

The remainder of the paper will be devoted to demonstrating the existence of solutions to (1) and (2). The paper will be organized according to the value of t, first assuming sufficiently large n.

Notes: The third and fifth authors worked on this question independently in the late 1990's; the results of sections 2 and 5 appeared in Sze's doctoral dissertation ([10]). The first and fourth authors were NSF-supported "Research Experience for Undergraduates" students of Sze's during Summer, 2002, and contributed to the results of sections 3, 4 and 7. The second author was an NSF-supported REU student of Ford's during 2002 and contributed to the results of section 6. The results of all computer calculations are available from the third author. Sze thanks Professor Ken Ono, his thesis advisor at Penn State, for his encouragement and advice.

2 Odd $t \geq 19$

In the case where $t \geq 19$ is odd (and n is sufficiently large), we can use the main theorem from Kiming's proof of the existence of t-cores ([11]) to in fact produce self-conjugate t-cores.

Proposition 2 *If $t \geq 19$ is an odd integer and $n > \frac{t(t^2-1)}{2} + 2(t-1)$, then n has a self-conjugate t-core partition.*

PROOF. First assume n is even. The condition on n implies that $\frac{n}{2} \geq \frac{t(t^2-1)}{4} + t - 1$. By applying the construction in the proof of the main theorem in [11] (the construction only requires that t be odd, not prime) to $\frac{n}{2}$, we obtain $x_1, \ldots, x_8$ satisfying both

$$\sum_{i=0}^{8} tx_i^2 + 2(i-1)x_i = n \qquad \text{and} \qquad \sum_{i=1}^{8} x_i = 0.$$

Combining these two equations and setting $x_i = 0$ for $i > 8$, we obtain $n = \sum_{i=1}^{(t-1)/2} tx_i^2 + 2ix_i$ – that is, a solution to equation (2).

If n is odd, then the condition on n implies that $\frac{n-1}{2} \geq \frac{t(t^2-1)}{4} + t - 1$, and then as above we obtain $x_1, \ldots, x_8$ such that $n - 1 = \sum_{i=1}^{8} tx_i^2 + 2ix_i$. Now setting $x_9 = \cdots = x_{(t-3)/2} = 0$ and $x_{(t-1)/2} = -1$, we obtain a solution to equation (2). □

3 $t = 13, 15, 17$

Proposition 3 *If $t \in \{13, 15, 17\}$ and $n \geq 6t(6t^2 - t + 1) + 23$, then n has a self-conjugate t-core partition.*

PROOF. First, assume n is not congruent to 2 (mod 3). Write $n = mt + r$ with $m \equiv 3 \pmod 6$ and $r < 6t$, so r is congruent to 0 or 1 (mod 3). The assumption $n \geq 6t(6t^2 - t + 1) + 23$ forces $m > r^2 - r$.

Because $r \equiv 0$ or $1 \pmod 3$, we have that $r^2 - r = r(r-1)$ is divisible by 3; also note that since m is odd and $r^2 - r$ is even, we have that $m - (r^2 - r)$ is odd. Thus $2(m - (r^2 - r))/3 \equiv 2 \pmod 4$. The form $a^2 + b^2 + 5c^2$ represents all such integers (by a method of [12, Chapter VIII], as outlined in [13]), so let a, b, c be integers such that

$$\frac{2(m - (r^2 - r))}{3} = a^2 + b^2 + 5c^2$$

(choosing a to be even if possible). If $2|a$, then b and c must have the same parity, and we set

$$\begin{aligned} x_1 &= \frac{a - 2b - 2c}{2} \\ x_2 &= \frac{-2a - b + c}{2} \\ x_3 &= \frac{a + 4c}{2} \\ x_4 &= \frac{b - 3c}{2} \\ x_{(t-1)/2} &= -r, \end{aligned}$$

all other $x_i = 0$. Then the integers x_i satisfy equation (2), and so n has a self-conjugate t-core partition.

If a is forced to be odd, then b must also be odd (else we would interchange a and b). Thus c must be even, and we set

$$\begin{aligned} x_1 &= \frac{-2a+3c}{2} \\ x_2 &= \frac{-a-b-4c}{2} \\ x_3 &= \frac{2b-c}{2} \\ x_4 &= \frac{a-b+2c}{2} \\ x_{(t-1)/2} &= -r, \end{aligned}$$

other $x_i = 0$. Then once again, the integers x_i satisfy equation (2), so n has a self-conjugate t-core partition.

If $n \equiv 2 \pmod 3$ and $t = 15$ or 17, then let $n' = n - 5$. Use the method above to choose the x_i corresponding to n', and set $x_{(t-5)/2} = 1$. Then the x_i satisfy equation (2).

If $n \equiv 2 \pmod 3$ and $t = 13$, then let $n' = n - 23$, choose x_i for n' as above, and set $x_5 = -1$. Again, this choice of x_i satisfies equation (2). $\square$

4 $t = 11$

Here we use yet another regular ternary form to establish

Proposition 4 *If n is an integer greater than 2, then n has a self-conjugate 11-core partition.*

PROOF. First assume $n \geq 4809$. Write $n = 11k' + r'$, with k' odd and $r' < 22$. Choose r and x_1 depending on r' according to the following list of ordered triples (r', r, x_1):

$$\begin{aligned} \{&(0,0,0),(1,265,-5),(2,1080,-10),(3,2445,-15),(4,4360,-20),\\ &(5,93,-3),(6,688,-8),(7,1833,-13),(8,3528,-18),(9,9,-1),\\ &(10,384,-6),(11,1309,-11),(12,2784,-16),(13,4809,-21),\\ &(14,168,-4),(15,873,-9),(16,2128,-14),(17,3933,-19),\\ &(18,40,-2),(19,525,-7),(20,1560,-12),(21,3145,-17)\} \end{aligned}$$

In each case, $r \equiv r' \pmod{22}$, so it is possible to write $n = 11k + r$ with k odd. The form $3a^2 + 2b^2 + c^2$ represents all odd positive integers (again by the method of [12]), so choose integers a, b, c such that $k = 3a^2 + 2b^2 + c^2$ and set

$$\begin{aligned} x_2 &= \frac{1}{3}(4a + 2b + c) \\ x_3 &= \frac{1}{3}(-3a + 3b) \\ x_4 &= \frac{1}{3}(-a - 2b + 2c) \\ x_5 &= \frac{1}{3}(a - b - 2c). \end{aligned}$$

Then we again have a solution to equation (2):

$$\begin{aligned} \sum_1^5 (11x_i^2 + 2ix_i) &= 11(3a^2 + 2b^2 + c^2) + 11x_1^2 + 2x_1 \\ &= 11k + r \\ &= n. \end{aligned}$$

This completes the proof under the assumption that $n \geq 4809$. A computer computation of self-conjugate 11-cores for $2 < n < 4809$ completes the proof of Theorem 1 for $t = 11$. $\square$

5 Even t

Here we use a generating function approach to establish the existence of t-cores for even $t \geq 8$.

We make use of the following generating functions for $t-$core partition and self-conjugate $t-$core partition numbers.

Let

$$\begin{aligned} c_t(n) &= \#\{\text{partitions } \lambda : |\lambda| = n \text{ and } \lambda \text{ is a } t-\text{core}\} \\ sc_t(n) &= \#\{\text{partitions } \lambda : |\lambda| = n \text{ and } \lambda \text{ is a self-conjugate } t-\text{core}\}. \end{aligned}$$

The generating functions for $c_t(n)$ and $sc_t(n)$ are below (see [9] and [7, Section 9] for the first; the second is obtained by specializing a more general identity of [15]).

$$\sum_{n=0}^{\infty} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{nt})^t}{1 - q^n}$$

$$\sum_{n=0}^{\infty} sc_t(n) \cdot q^n = \begin{cases} \prod_{n=1}^{\infty} (1-q^{2tn})^{\frac{t-1}{2}} \cdot \frac{1+q^{2n-1}}{1+q^{t(2n-1)}} & \text{if } t \text{ is odd,} \\ \prod_{n=1}^{\infty} (1-q^{2tn})^{\frac{t}{2}} \cdot (1+q^{2n-1}) & \text{if } t \text{ is even,} \end{cases}$$

For convenience, we will denote the generating function for self-conjugate $t-$cores by

$$\phi_t(q) = \sum_{n=0}^{\infty} sc_t(n) \cdot q^n.$$

Let $\theta(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)}$, and define $\Delta_n = \frac{n(n+1)}{2}$. Then the following equality is a classical consequence of the Jacobi Triple Product Identity:

$$\theta(q) = \sum_{n=0}^{\infty} q^{\Delta_n} \tag{3}$$

It will be useful to factor $\phi_t(q)$ in the following ways.

Lemma 5 *(1) If $t = 2t'$ with t' even, then*

$$\phi_t(q) = \phi_{t'}(q) \cdot \theta\left(q^t\right)^{\frac{t'}{2}}.$$

(2) If $t = 2t'$ with t' odd, then

$$\phi_t(q) = \phi_{t'}(q) \cdot \theta\left(q^t\right)^{\frac{t'-1}{2}} \theta\left(q^{t'}\right).$$

(3) If $t = 2^k t'$, t' is odd, and $1 \leq i < k$, then

$$\phi_t(q) = \phi_{2^i t'}(q) \cdot \prod_{j=i+1}^{k} \theta\left(q^{2^j t'}\right)^{2^{j-1} t'}.$$

PROOF.

(1) If $t = 2t'$, where t' is even, then

$$\begin{aligned} \phi_t(q) &= \prod_{n=1}^{\infty} (1-q^{2tn})^{t'} (1+q^{2n-1}) \\ &= \prod_{n=1}^{\infty} \left(\frac{(1-q^{2tn})^2}{(1-q^{tn})} \right)^{\frac{t'}{2}} \prod_{n=1}^{\infty} (1-q^{tn})^{\frac{t'}{2}} (1+q^{2n-1}) \\ &= \left(\theta\left(q^t\right)\right)^{\frac{t'}{2}} \phi_{t'}(q). \end{aligned}$$

(2) If $t = 2t'$where t' is odd, then

$$\begin{aligned}
\phi_t(q) &= \prod_{n=1}^{\infty}(1-q^{2tn})^{t/2}(1+q^{2n-1}) \\
&= \prod_{n=1}^{\infty}\left(\frac{(1-q^{2tn})^2}{1-q^{tn}}\right)^{\frac{t-2}{4}} \cdot \frac{(1-q^{tn})^2}{1-q^{t'n}} \cdot \left(1+q^{2n-1}\right)(1-q^{tn})^{\frac{t-2}{4}} \cdot \frac{\frac{1-q^{2tn}}{1-q^{tn}}}{\frac{1-q^{tn}}{1-q^{t'n}}} \\
&= \theta\left(q^t\right)^{\frac{t'-1}{2}} \cdot \theta\left(q^{t'}\right) \cdot \prod_{n=1}^{\infty}(1+q^{2n-1})(1-q^{2t'n})^{\frac{t'-1}{2}}\frac{\left(1+q^{2t'n}\right)}{(1+q^{t'n})} \\
&= \theta\left(q^t\right)^{\frac{t'-1}{2}} \cdot \theta\left(q^{t'}\right) \cdot \prod_{n=1}^{\infty}\left(1-q^{2t'n}\right)^{(t'-1)/2} \cdot \frac{1+q^{2n-1}}{1+q^{t'(2n-1)}} \\
&= \theta\left(q^t\right)^{\frac{t'-1}{2}} \cdot \theta\left(q^{t'}\right) \cdot \phi_{t'}(q).
\end{aligned}$$

(3) The third identity follows by successively applying parts 1 and 2.

□

Application of the above Lemma shows that following factorization of $\phi_{2t}(q)$ always exists

$$\phi_{2t}(q) = \phi_t(n) \cdot \sum_{n=0}^{\infty} a(n)\, q^{tn}$$

where $a(n) \geq 0$, and $a(0) = 1$. Comparing coefficients of q^n, it follows that $sc_{2t}(n) \geq sc_t(n)$ always holds (of course, this is clear from the definitions as any t-core of n is also a $2t$-core of n).

The hooks of the main diagonal nodes of a Ferrers-Young diagram partition that diagram. That the main diagonal hook numbers of self-conjugate partitions are odd and distinct is known as *Euler's Partition Theorem*. In particular, let $sc(n)$ denote the number of self-conjugate partitions of n and let

$$p_d(n, S) := \#\{\text{partitions of } n \text{ into distinct parts taken from } S\}$$

where S is any subset of the natural numbers; then

$$sc(n) = p_d(n,\ 2\mathbb{Z}^+ + 1).$$

Example 6 *The self-conjugate partition* $\Lambda = (7,5,4,3,2,1,1)$ *has main diagonal hook numbers* $(13,7,3)$ *which form a partition of* 23 *with distinct odd numbers.*

We now study the existence self-conjugate $t-$cores with bounded main diagonal hook numbers.

Lemma 7 *When $m \geq 3$, $p_d(n, \{1, 3, ..2m-1\}) > 0$ if and only if $n \in \{0, 1\} \cup [3, m^2-3] \cup \{m^2\}$.*

PROOF. By induction on m. By inspection, $p_d(n, \{1, 3, 5\}) > 0$ if and only if $n \in \{0, 1\} \cup [3, 6] \cup \{9\}$. Now assume that $m \geq 3$ and $p_d(n, \{1, 3, \ldots, 2m-1\}) > 0$ exactly for $n \in \{0, 1\} \cup [3, m^2-3] \cup \{m^2\}$. Note that

$$\begin{aligned} &p_d(n, \{1, 3, \ldots, 2m+1\}) \\ &= p_d(n, \{1, 3, \ldots, 2m-1\}) + p_d(n-(2m+1), \{1, 3, \ldots, 2m-1\}) \end{aligned}$$

Thus, $p_d(n, \{1, 3, \ldots, 2m+1\}) > 0$ precisely when

$$\begin{aligned} n &\in \{0, 1\} \cup \left[3, m^2-3\right] \cup \left\{m^2\right\} \\ &\cup \{2m+1, 2m+2\} \cup \left[2m+4, m^2+2m-2\right] \cup \left\{(m+1)^2\right\} \\ &= \{0, 1\} \cup \left[3, m^2+2m-2\right] \cup \left\{(m+1)^2\right\}. \end{aligned} \tag{4}$$

□

Proposition 8 *For $m \geq 3$, $t > 2m-1$, there exists a self-conjugate $t-$core partition Λ of n where $n \in \{0, 1\} \cup [3, m^2-3] \cup \{m^2\}$ and where the main diagonal hooks of Λ are distinct odd numbers no bigger than $2m-1$.*

PROOF. If the largest hook number of a self-conjugate partition Λ is $2m-1$, then Λ is trivially a $t-$ core for all $t > 2m-1$. The main diagonal hook numbers of such a partition form a subset of the odd integers from 1 to $2m-1$. The prior lemma now proves the Proposition. □

We are now in a position to completely prove the positivity of the even cases.

Proposition 9 *If $t' \geq 5$ is odd, then $sc_{2t'}(n) > 0$ for all $n \neq 2$.*

PROOF. By Lemma 5(2) we have

$$\begin{aligned} \phi_{2t'}(q) &= \theta\left(q^{2t'}\right)^{\frac{t'-1}{2}} \cdot \theta\left(q^{t'}\right) \cdot \phi_{t'}(q) \\ &= \theta\left(q^{2t'}\right)^{\frac{t'-5}{2}} \cdot \theta\left(q^{2t'}\right)^2 \theta\left(q^{t'}\right) \cdot \phi_{t'}(q) \end{aligned}$$

Note that

$$\theta\left(q^{2t'}\right)^2 \theta\left(q^{t'}\right) = \prod_{n=1}^{\infty} \frac{\left(1-q^{2t'n}\right)^4}{\left(1-q^{t'n}\right)} = \sum_{n=0}^{\infty} c_4(n)\, q^{t'n}.$$

It is well known that $c_4(n) \geq 1$ for all $n \geq 0$. In fact, $c_4(n) = \frac{1}{2}h(-32n-20)$ where $h(N)$ is the order of the class group of discriminant N binary quadratic forms [17].

Now since the coefficients in the Fourier expansion of $\theta\left(q^{2t'}\right)^{\frac{t'-5}{2}}$ are all non-negative, and the constant term is 1, we see that the coefficients of $q^{t'n}$ in the Fourier expansion of

$$\theta\left(q^{2t'}\right)^{\frac{t'-5}{2}}\theta\left(q^{2t'}\right)^{2}\theta\left(q^{t'}\right)$$

are strictly positive. Thus, it suffices to show $sc_{t'}(n) > 0$ for $n \in \{0,1\} \cup [3,\dots t'-1] \cup \{t'+2\}$. When $t' \geq 7$, this is demonstrated by setting $m = \frac{t'-1}{2}$ in Proposition 8. For the case $t' = 5$, it can be explicitly checked that $sc_5(0) = sc_5(1) = sc_5(3) = sc_5(4) = sc_5(7) = 1 > 0$. □

Proposition 10 *When $t > 0$ is even, $sc_t(n) > 0$ for all $n \neq 2$ if and only if $t \geq 8$.*

PROOF. Let $t = 2^k t'$ with $k \geq 1$ and t' odd. If $t' \geq 5$, then by Proposition 9 $sc_t(n) \geq sc_{2t'}(n) > 0$ for all $n \neq 2$. It now remains to examine the two cases $t' = 3$, and $t' = 1$.

If $t = 12$, then by Lemma 5,

$$\phi_{12}(n) = \theta\left(q^{12}\right)^{6}\theta\left(q^{6}\right)\theta\left(q^{3}\right)\phi_3(q).$$

Since $\theta(q)^3 = \sum_{n=0}^{\infty} a_n q^n$, where

$$a_n = \#\{(x,y,z) : x,y,z \geq 0 \text{ and } \Delta_x + \Delta_y + \Delta_z = n\}$$

then by Gauss's Eureka theorem (every positive integer is the sum of at most three triangular numbers), the Fourier series of $\theta(q^{12})^3$ has positive coefficients for each term whose exponent is a multiple of 12. It follows that the Fourier series expansion of $\theta(q^{12})^6$ has positive coefficients for each term whose exponent is a multiple of 12. Note that the remaining factor of $\phi_{12}(q)$ has expansion

$$\theta\left(q^{6}\right)\theta\left(q^{3}\right)\phi_3(q) = 1+q+q^3+q^4+q^5+q^6+q^7+2q^8+2q^9+2q^{10}+2q^{11}+3q^{14}\dots.$$

Hence, $sc_{12}(n) > 0$ for all $n \neq 2$. Furthermore $sc_{3\cdot 2^k}(n) \geq sc_{12}(n) > 0$ for all $k \geq 2$.

Finally, if $t = 8$, then by Lemma 5

$$\begin{aligned}\phi_8(q) &= \theta\left(q^8\right)^4\theta\left(q^4\right)^2\phi_2(q)\\ &= \left(\sum_{n=0}^{\infty}c_4(n)\,q^{4n}\right)^2\phi_2(q)\,.\end{aligned}$$

Since $c_4(n) > 0$ for all n, and $\phi_2(q) = 1 + q + q^3 + q^6 + \dots$, it follows that $sc_8(n) > 0$ for all non-negative $n \neq 2$. For the remaining cases of $t = 2^k$, we have $sc_{2^k}(n) \geq sc_8(n) > 0$ for all $k \geq 3$.

There are no other even t for which $sc_t(n) > 0$ for all non-negative $n \neq 2$ since $sc_2(4) = sc_4(8) = sc_6(13) = 0$. □

6 Small n

The proofs of sections 2 and 3 required n to be larger than certain bounds. Here we fill in the gaps to establish Theorem 1. The approach is to use three of the x_i to obtain "most" of n (using Gauss's three squares theorem), and then use the remaining x_i to find a solution to (2).

Proposition 11 *If $t \geq 29$ is odd and $n \leq U(t) := \frac{1}{400}t(t^2 - 26t + 37)^2$, then equation (2) has an integral solution.*

PROOF. We begin with two lemmas:

Lemma 12 *If $t \geq 29$ and $n < U(t)$, then there exist integers x_1, x_2, x_3 such that*

$$n - \sum_{i=1}^{3}(tx_i^2 + 2ix_i) < E(t) := (\frac{t-1}{2} - 3)(\frac{t-1}{2} - 4) + (\frac{t-1}{2} - 6).$$

PROOF. Let $t \geq 29$ be prime, and $n < U(t)$. Write $n = kt+j$ with $0 \leq j < 3t$ and $k \equiv 1$ or $2 \pmod 4$. Then by Gauss's three squares theorem, there exist positive integers $a \geq b \geq c$ such that $k = a^2 + b^2 + c^2$. Because $a \geq b \geq c$ and $0 \leq j$, we obtain upper bounds for a, b, and c:

$$\begin{aligned}a &\leq \sqrt{k} = \sqrt{\frac{n-j}{t}} \leq \sqrt{\frac{n}{t}}\\ b &\leq \sqrt{\frac{k}{2}} = \sqrt{\frac{n-j}{2t}} \leq \sqrt{\frac{n}{2t}}\\ c &\leq \sqrt{\frac{k}{3}} = \sqrt{\frac{n-j}{3t}} \leq \sqrt{\frac{n}{3t}}\end{aligned} \qquad (5)$$

Now let $x_1 = a, x_2 = b$, and $x_3 = -c$. Then

$$\begin{aligned}
n - \sum_{i=1}^{3}(tx_i^2 + 2ix_i) &= n - (t(a^2 + b^2 + c^2) - 2a - 4b + 6c) \\
&= n - (pk - 2a - 4b + 6c) \\
&= n - ((n - j) - 2a - 4b + 6c) && \text{since } n = kt + j \\
&= j + 2a + 4b - 6c \\
&\le j + 2a + 4b \\
&\le 3t + 2\sqrt{\frac{n}{t}} + 4\sqrt{\frac{n}{2t}} && \text{by (5)} \\
&= 3t + (2 + 2\sqrt{2})\sqrt{\frac{n}{t}} \\
&< 3t + 5\sqrt{\frac{n}{t}}
\end{aligned}$$

Now $n < U(t)$ implies that

$$3t + 5\sqrt{\frac{n}{t}} < E(t)$$

if $t^2 - 26t + 37 \ge 0$. Since $t \ge 29$, this last inequality is satisfied and we have

$$n - \sum_{i=1}^{3}(tx_i^2 - 2ix_i) < 3t + 5\sqrt{\frac{n}{t}} < E(t).$$

□

Lemma 13 *Let m be a positive integer. If $n \ne 2$ and*

$$0 \le n \le \begin{cases} 1 & \text{if } m = 1 \\ 4 & \text{if } m = 2 \\ m(m-1) + (m-3) & \text{if } m \ge 3 \end{cases},$$

then (2) has a solution with $x_1 = x_2 = \cdots = x_{(t-1)/2-m} = 0$ and $x_i \in \{0, -1\}$ for $i > \frac{t-1}{2} - m$.

PROOF. Let $\frac{t-1}{2} \ge i > \frac{t-1}{2} - m$. Then $i = \frac{p-1}{2} + 1 - j$ for some $1 \le j \le m$. If $x_i = -1$, then $px_i^2 + 2ix_i = 2j - 1$, the jth odd positive integer. So to prove the Lemma, it suffices to prove that if n satisfies the conditions, then n can be represented as a sum of some subcollection of the first m positive odd integers.

For $m = 1, 2$, this is accomplished by inspection; for $m \ge 3$, it follows from Lemma 7. □

Recall that $n < U(t)$, and so Lemma 12 establishes the existence of x_1, x_2, and x_3 such that

$$n - \sum_{i=1}^{3}(tx_i^2 + 2ix_i) < E(t).$$

Now Lemma 13 assures us that there exist $x_4, x_5, \ldots, x_{(t-1)/2} \in \{0, -1\}$ such that

$$n - \sum_{i=1}^{3}(tx_i^2 + 2ix_i) = \sum_{i=4}^{\frac{t-1}{2}}(tx_i^2 + 2ix_i).$$

Then

$$n = \sum_{i=1}^{3}(tx_i^2 + 2ix_i) + \sum_{i=4}^{\frac{t-1}{2}}(tx_i^2 + 2ix_i) = \sum_{i=1}^{\frac{t-1}{2}}(tx_i^2 + 2ix_i).$$

This completes the proof of the proposition. $\square$

For odd $t \geq 41$, $U(t) > \frac{t(t^2-1)}{2} + 2(t-1)$, the bound required for Proposition 2. Thus Propositions 2 and 11 establish Theorem 1 for odd $t \geq 41$.

For odd t with $13 \leq t \leq 39$, we still need to demonstrate the existence of self-conjugate t-cores for small n (less than the bounds required in Propositions 2 and 3). This has been accomplished by a computer implementation of a greedy algorithm to choose $x_1, \ldots, x_j$ (for some j) in such a way that

$$n - \sum_{i=1}^{j}(tx_i^2 + 2ix_i)$$

satisfies the bounds of Lemma 13, which then guarantees that $x_{j+1}, \ldots, x_{(t-1)/2}$ exist to give a solution to equation (2).

7 $t = 9$

Finally, we discuss self-conjugate 9-core partitions. An experimental check indicates that numbers of the form $n = (4^k - 10)/3$ are the only n in the first 50000 that have no such partitions. Therefore, we conjecture that all such n are of this form. Here we prove that no number n of this form has a 9-core partition. *Note:* We have recently been sent a proof of the converse by Peter Montgomery.

Proposition 14 *If $n = \frac{4^k-10}{3}$ for some positive integer k, then n has no self-conjugate 9-core partition.*

PROOF. Suppose otherwise, that for some k, the number $(4^k - 10)/3$ has a 9-core partition. Then, by (2) we have

$$\frac{4^k - 10}{3} = 9x_1^2 + 2x_1 + 9x_2^2 + 4x_2 + 9x_3^2 + 6x_3 + 9x_4^2 + 8x_4.$$

Completing the square, this becomes

$$3 \cdot 4^k = (9x_1 + 1)^2 + (9x_2 + 2)^2 + (9x_3 + 3)^2 + (9x_4 + 4)^2. \tag{6}$$

Let k be the smallest positive integer for which a solution to (6) exists. By inspection, $k > 2$. Thus the left hand side of (6) is divisible by 8; this forces each term on the right hand side of (6) to be even. Thus x_1 and x_3 are odd; x_2 and x_4 are even. Set

$$X_4 = -\frac{x_1 + 1}{2} \qquad X_1 = \frac{x_2}{2} \qquad X_3 = -\frac{x_3 + 1}{2} \qquad X_2 = \frac{x_4}{2}$$

Now when we evaluate the right hand side of (6) using the new X_i, we find

$$(9X_1 + 1)^2 + (9X_2 + 2)^2 + (9X_3 + 3)^2 + (9X_4 + 4)^2 = \frac{1}{4}((9x_1 + 1)^2 + (9x_2 + 2)^2 + (9x_3 + 3)^2 + (9x_4 + 4)^2) = 3 \cdot 4^{k-1},$$

which contradicts the minimality of k. Thus no number of the form $3 \cdot 4^k$ has a self-conjugate 9-core partition. □

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