

IRREDUCIBLE RESTRICTIONS OF REPRESENTATIONS
OF THE SYMMETRIC GROUPS

BEN FORD

ABSTRACT

A proof is given of a recent conjecture of Jantzen and Seitz giving a necessary and sufficient condition for a representation of the symmetric group on n objects (over an algebraically closed field of prime characteristic $p < n$) to remain irreducible upon restriction to the symmetric group on $n - 1$ objects.

In this paper Sym_n denotes the symmetric group on n objects, and Sym_{n-1} is embedded in Sym_n as the stabilizer of one of the objects. Representations are understood to be over an algebraically closed field K of characteristic $p > 0$. The methods used here were originally developed in work on the restriction of representations of algebraic groups; see [0] and [0].

We recall a few facts from the representation theory of the symmetric groups (see [0] for more). The irreducible $K\text{Sym}_n$ -modules are parameterized by p -regular partitions $\mu = (\mu_1^{m_1}, \dots, \mu_l^{m_l})$ of n , where p -regular means that $m_i \leq p - 1$ for every $1 \leq i \leq l$.

A fundamental result in the representation theory of the symmetric groups over a field of characteristic 0 is the branching theorem, which describes exactly what happens when an irreducible representation of Sym_n is restricted to Sym_{n-1} . No

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similar result is known for representations in finite characteristic $p < n$, and finding branching theorems over fields of arbitrary characteristic was the object of [0]. A more limited problem is determining those representations of $KSym_n$ which remain irreducible for $KSym_{n-1}$. In [0], Jantzen and Seitz state the following conjecture, proving the “if” direction (for $p = 2$ the conjecture was made by D. Benson):

CONJECTURE 1 *Let M be an irreducible $KSym_n$ -module corresponding to the p -regular partition $\mu = (\mu_1^{m_1}, \dots, \mu_l^{m_l})$. Then $M|_{Sym_{n-1}}$ is irreducible if and only if $\mu_i - \mu_{i+1} + m_i + m_{i+1} \equiv 0 \pmod{p}$, for $i = 1, \dots, l - 1$.*

Note: Kleshchev gives a different proof of this conjecture in [0]. His proof consists of two pieces; the first, his Corollary 2.11, is quoted below (Proposition 3) and is the easier of the two. The second part (Theorem 7.2 in [0], showing — in the notation used following Proposition 3 below — that if V_1 is an irreducible L' -module, then the congruences on the coefficients of λ hold) requires a long computational proof, with considerable attention paid to the exact values of many constants. As the proof presented here is considerably shorter and, we believe, more transparent, it seemed worthwhile to present it.

This work was mentioned in the original paper of Jantzen and Seitz ([0]), which motivated Kleshchev’s work, as going “a long way towards establishing the conjecture.” In fact, all that was missing was Corollary 2.11 of [0].

One well-known connection between the representation theory of the symmetric groups and that of the general linear groups is the Schur functor: Starting with a representation of $GL_n(K)$, one gets an action of Sym_n on the weight space corresponding to the determinant character. Thus, a map from rational representations of $GL_n(K)$ to K -representations of Sym_n is obtained. Jantzen and Seitz exploit this correspondence to prove that Conjecture 1 is implied by Theorem 2 below.

Let $Y = SL_n(K)$, and let $\{\alpha_1, \dots, \alpha_{n-1}\} = \Pi(Y)$ be the set of fundamental roots of the root system $\Sigma(Y) = \Sigma$ (with respect to a fixed Borel subgroup B and maximal torus T). The fundamental dominant weights of T will be denoted

by $\lambda_1, \dots, \lambda_{n-1}$, with $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n-1$. Let $V = V(\lambda)$ be a non-trivial irreducible restricted high weight module for Y of highest weight $\lambda = a_1\lambda_1 + a_2\lambda_2 + \dots + a_{n-1}\lambda_{n-1}$, and let a_{i_1}, \dots, a_{i_m} be the non-zero a_i , with $i_1 < i_2 < \dots < i_m = k$. Finally, P will be the maximal parabolic subgroup of Y corresponding to $\alpha_{n-1} \in \Pi(Y)$ and $P = QL$ its Levi decomposition (with respect to T), so $L' \cong SL_{n-1}$. If μ is a weight of T , then V_μ is the weight space in V of weight μ .

Let $V_1 = \bigoplus V_\mu$, with the sum taken over those $\mu = \lambda - (b_1\alpha_1 + b_2\alpha_2 + \dots + b_{n-1}\alpha_{n-1})$ with $b_{n-1} = 1$. Let v^+ be a non-zero T -high weight vector in V .

We now have enough notation to state the main result, which implies Conjecture 1:

THEOREM 2 *V_1 has exactly one restricted L' -composition factor if and only if*

$$a_{i_j} + a_{i_{j+1}} + i_{j+1} - i_j \equiv 0 \pmod{p}$$

for all $j = 1, \dots, m-1$.

PROOF: The “if” direction of this was proved in [0]. In [0], Kleshchev proves

PROPOSITION 3 *V_1 has exactly one restricted L' -composition factor if and only if V_1 is an irreducible L' -module.*

So our task is to show that if V_1 is an irreducible L' -module, then the congruences on the coefficients of λ hold.

Since we are assuming that V is a restricted Y -module, we may work over the Lie algebra instead of the group. We use the standard basis $\{e_\alpha, f_\alpha, h_i | \alpha \in \Sigma^+(Y), 1 \leq i \leq n-1\}$ for the Lie algebra of Y , and we let $V_{i,l}$ be the subspace of $V_{\lambda - (\alpha_i + \dots + \alpha_l)}$ spanned by all elements other than $f_{\alpha_i + \dots + \alpha_l} v^+$ of the form

$$f_{\alpha_i + \dots + \alpha_j} f_{\alpha_{j+1} + \dots + \alpha_{j'}} \cdots f_{\alpha_{j^{(m)}+1} + \dots + \alpha_l} v^+, \quad (\dagger)$$

with $i < j < \dots < j^{(m)} < l$. To ease the notational pain, we write $f_{[i,j]}$ for $f_{\alpha_i + \dots + \alpha_j}$.

The L' -module V_1 has a filtration

$$\begin{aligned} 0 &\leq \langle f_{[k,n-1]}v^+ \rangle \leq \langle f_{[k,n-1]}v^+, f_{[k-1,n-1]}v^+ \rangle \\ &\leq \langle f_{[k,n-1]}v^+, f_{[k-1,n-1]}v^+, f_{[k-2,n-1]}v^+ \rangle \\ &\leq \cdots \leq V_1, \end{aligned}$$

where a_k is the “last” non-zero coefficient of λ , i.e. $a_l = 0$ for every $l > k$. Since V_1 is irreducible as an L' -module, this filtration has only one member, i.e. $f_{[i,n-1]}v^+ \in \langle f_{[k,n-1]}v^+ \rangle$ for all $i < k$. In particular, $f_{[i,n-1]}v^+ \in V_{i,n-1}$ for all $i < k$.

LEMMA 4 *Let a_k be the last non-zero coefficient of λ . If $i < k$, then $f_{[i,k]}v^+ \in V_{i,k}$.*

PROOF: To say that $f_{[i,n-1]}v^+ \in V_{i,n-1}$ is to say that there is a relation

$$f_{[i,n-1]}v^+ = \sum a_{(j,j',\dots)} f_{[i,j]} f_{[j+1,j']} \cdots f_{[j^{(m)},n-1]}v^+, \quad (*)$$

for some constants $a_{(j,j',\dots)}$, with $m \geq 2$ for every summand and $i < j < j' < \cdots < j^{(m)} \leq k$. Notice that for any $l < k$, $f_{[l,n-1]}v^+ = a f_{[k+1,n-1]} f_{[l,k]}v^+$ for some non-zero $a \in K$, as $f_{[k+1,n-1]}v^+ = 0$. Similarly, for any $l > k$, $f_{[l,n-1]}v^+ = 0$. So (*) may be rewritten as

$$f_{[k+1,n-1]} f_{[i,k]}v^+ = f_{[k+1,n-1]} \sum b_{(j,j',\dots)} f_{[i,j]} \cdots f_{[j^{(m)},k]}v^+$$

for some constants $b_{(j,j',\dots)}$. We apply $e_{\alpha_{k+1}+\cdots+\alpha_{n-1}}$ and see that this in turn implies

$$f_{[i,k]}v^+ = \sum c_{(j,j',\dots)} f_{[i,j]} \cdots f_{[j^{(m)},n-1]}v^+$$

for some constants $c_{(j,j',\dots)}$, since $(\lambda - (\alpha_i + \cdots + \alpha_k))(h_{\alpha_{k+1}+\cdots+\alpha_{n-1}}) \neq 0$. \diamond

PROPOSITION 5 *Let V and λ be as above. Let a_i, a_m be non-zero coefficients with $m > i$. If $f_{[r,m]}v^+ \in V_{r,m}$ for all r with $i \leq r < m$, then $f_{[i,j]}v^+ \in V_{i,j}$, where a_j is the first non-zero label after a_i .*

PROOF: Assume the hypotheses. If $j = m$, the Proposition is vacuous; so assume there are non-zero coefficients between a_i and a_m ; we proceed by induction on the number of such non-zero coefficients. Let a_l be the last non-zero coefficient before a_m .

For the calculations below, recall that since v^+ is a maximal vector, $e_\beta v^+ = 0$ for any $\beta \in \Sigma^+$, and that e_β and f_δ commute whenever $\beta - \delta \notin \Sigma$. If $\beta - \delta \in \Sigma$, then $e_\beta f_\delta = f_\delta e_\beta + d e_{\beta-\delta}$ for some $d = N(\beta, -\delta) \in \mathbb{Z}$ (where $e_{-\alpha} = f_\alpha$ for $\alpha \in \Sigma^+$). In our case ($Y = SL_n$), $d = \pm 1$.

We require the following lemma to complete the proof of the Proposition.

LEMMA 6 *If a_i , a_l , and a_m are non-zero coefficients of λ with $i < l < m$ and a_l is the last non-zero coefficient before a_m , then $f_{[r,m]}v^+ \in V_{r,m}$ for all r with $i \leq r < m$ implies that $f_{[r,l]}v^+ \in V_{r,l}$ for all r with $i \leq r < l$.*

PROOF: Assume that there is an $r \in [i, l-1]$ such that $f_{[r,l]}v^+ \notin V_{r,l}$.

The assumption that $f_{[r,m]}v^+ \in V_{r,m}$ implies that there is some non-zero linear combination

$$0 = f_{[r,m]}v^+ + \sum_{(\rho_1, \dots, \rho_s)} d_{(\rho_1, \dots, \rho_s)} f_{\rho_1} \cdots f_{\rho_s} v^+, \quad (*)$$

where $f_{\rho_1} \cdots f_{\rho_s} v^+$ is a term of type (\dagger) , with $s \geq 2$ and $\sum \rho_j = \alpha_r + \cdots + \alpha_m$.

Now we apply some $e_{\gamma_1} \cdots e_{\gamma_q}$ to $(*)$, with $\gamma_1 = \alpha_{l+1} + \cdots + \alpha_l$, $\gamma_2 = \alpha_{l+1} + \cdots + \alpha_{l'}$, ..., and $\sum \gamma_j = \alpha_{l+1} + \cdots + \alpha_m$. Notice that such a product of e_γ 's applied to a summand of $(*)$ gives a multiple of a generator for $V_{\lambda - (\alpha_r + \cdots + \alpha_l)}$ of type (\dagger) . Consider those summands which give a multiple of $f_{[r,l]}v^+$. Since each f_{ρ_a} must involve some α_q with $a_q \neq 0$, these terms are exactly the

$$F_j v^+ = f_{[r, l+j]} f_{[l+j+1, m]} v^+, \quad 0 \leq j \leq m-l,$$

since all the coefficients between a_l and a_m are 0. Notice that $j = m-l$ gives $F_{m-l} v^+ = f_{[r,m]} v^+$.

Let the coefficient of $F_j v^+$ in $(*)$ be b_j ($= d_{(\alpha_r + \cdots + \alpha_{l+j}, \alpha_{l+j+1} + \cdots + \alpha_m)}$); note that $b_{m-l} = 1$. Since by our assumption $f_{[r,l]}v^+$ is not a combination of other terms

(†) which appear in the $\lambda - (\alpha_r + \cdots + \alpha_l)$ -weight space, any $e_{\gamma_1} \cdots e_{\gamma_q}$ (with the γ_a 's as above) must kill $\sum b_j F_j v^+$. In particular,

$$E_s = e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} e_{\alpha_{l+s+1} + \cdots + \alpha_m}, \quad 0 \leq s < m - l,$$

must kill this sum.

Now assume $f_{[r, l-1]} v^+ \notin V_{r, l-1}$. Consider the summands in (*) which give a multiple of $f_{[r, l-1]} v^+$ when some $e_{\gamma_1} \cdots e_{\gamma_q}$ (where $\gamma_j \succ 0$ and $\sum \gamma_j = \alpha_l + \cdots + \alpha_m$) is applied. They are the $F_j v^+$ ($0 \leq j \leq m - l$), and

$$G_j v^+ = f_{[r, l-1]} f_{[l, l+j]} f_{[l+j+1, m]} v^+, \quad 0 \leq j < m - l$$

(note that we need not include $G_{m-l} = f_{[r, l-1]} f_{[l, m]} v^+$ here by our assumption that $f_{[l, m]} v^+$ is a consequence of the terms already listed).

By the assumption that $f_{[r, l-1]} v^+ \notin V_{r, l-1}$, it follows that $f_{[r, l-1]} v^+ \neq 0$. So any $e_{\gamma_1} \cdots e_{\gamma_q}$ (as above) must kill

$$\sum_{j=0}^{m-l} b_j F_j v^+ + \sum_{j=0}^{m-l-1} c_j G_j v^+,$$

where c_j is the coefficient of G_j in (*). In particular, $e_{\alpha_l} E_s$ must kill $\sum c_j G_j v^+$ for every $0 \leq s < m - l$. The following argument shows that $E_s G_j v^+ = 0$ if $s \neq j$: Assume $s < j$. Then

$$\begin{aligned} E_s G_j v^+ &= (e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} e_{\alpha_{l+s+1} + \cdots + \alpha_m} f_{[r, l-1]} f_{[l, l+j]} f_{[l+j+1, m]}) v^+ \\ &= e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} (f_{[r, l-1]} f_{[l, l+j]} e_{\alpha_{l+s+1} + \cdots + \alpha_m} f_{[l+j+1, m]} v^+) \\ &= \pm e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} (f_{[r, l-1]} f_{[l, l+j]} e_{\alpha_{l+s+1} + \cdots + \alpha_{l+j}} v^+) = 0. \end{aligned}$$

A similar calculation holds for $s > j$. Also,

$$\begin{aligned} e_{\alpha_l} E_j G_j v^+ &= e_{\alpha_l} f_{[r, l-1]} e_{\alpha_{l+1} + \cdots + \alpha_{l+j}} f_{[l, l+j]} e_{\alpha_{l+j+1} + \cdots + \alpha_m} f_{[l+j+1, m]} v^+ \\ &= \pm e_{\alpha_l} f_{[r, l-1]} f_{\alpha_l} h_{\alpha_{l+j+1} + \cdots + \alpha_m} v^+ \\ &= \pm a_l a_m f_{[r, l-1]} v^+. \end{aligned}$$

So:

$$0 = e_{\alpha_l} E_0(\sum c_j G_j v^+) = \pm c_0 a_l a_m f_{[r, l-1]} v^+ \Rightarrow c_0 = 0$$

$$0 = e_{\alpha_l} E_1(\sum c_j G_j v^+) = \pm c_1 a_l a_m f_{[r, l-1]} v^+ \Rightarrow c_1 = 0$$

\vdots

$$0 = e_{\alpha_l} E_{m-l-1}(\sum c_j G_j v^+) = \pm c_{m-l-1} a_l a_m f_{[r, l-1]} v^+ \Rightarrow c_{m-l-1} = 0$$

So this implies $\sum c_j G_j v^+ = 0$. But then:

$$\begin{aligned} 0 &= e_{\alpha_l + \dots + \alpha_m} (\sum b_j F_j v^+ + \sum c_j G_j v^+) = e_{\alpha_l + \dots + \alpha_m} (\sum b_j F_j v^+) \\ &= \pm f_{[r, l-1]} v^+ \end{aligned}$$

(as $F_{m-l} v^+ = f_{[r, m]} v^+$ is the only term in $\sum b_j F_j v^+$ not killed by $e_{\alpha_l + \dots + \alpha_m}$). But this is a contradiction.

So our assumption that there was an r , $i \leq r < l$, such that $f_{[r, l]} v^+ \notin V_{r, l}$ and $f_{[r, l-1]} v^+ \notin V_{r, l-1}$ must be false; i.e. for every r , $i \leq r < l$, either (i) $f_{[r, l]} v^+ \in V_{r, l}$, or (ii) $f_{[r, l-1]} v^+ \in V_{r, l-1}$. We want to show that in fact (i) holds always. There are several cases:

(a) If $r < l - 1$ and $f_{[r, l-1]} v^+ \in V_{r, l-1}$, we see that:

$$\begin{aligned} f_{[r, l]} v^+ &= f_{\alpha_l} f_{[r, l-1]} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \\ &= \sum_{(\rho_1, \dots, \rho_j), j \geq 2} f_{\alpha_l} f_{\rho_1} \cdots f_{\rho_j} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \\ &= \sum f_{\rho_1} \cdots f_{\rho_{j-1}} f_{\alpha_l} f_{\rho_j} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \\ &= \sum f_{\rho_1} \cdots f_{\rho_j} f_{\alpha_l} v^+ + s f_{\rho_1} \cdots f_{\rho_j + \alpha_l} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \end{aligned}$$

for some integers s (depending on (ρ_1, \dots, ρ_j)), so $f_{[r, l]} v^+ \in V_{r, l}$.

(b) If $r = l - 1$ and $a_{l-1} \neq 0$, (ii) cannot happen, since $V_{l-1, l-1} = 0$ and $f_{[r, l-1]} v^+ = f_{\alpha_{l-1}} v^+ \neq 0$.

(c) If $r = l - 1$ and $a_{l-1} = 0$, then

$$f_{\alpha_{l-1} + \alpha_l} v^+ = \pm (f_{\alpha_l} f_{\alpha_{l-1}} v^+ - f_{\alpha_{l-1}} f_{\alpha_l} v^+) = \pm f_{\alpha_{l-1}} f_{\alpha_l} v^+,$$

since $a_{l-1} = 0$. So $f_{[r,l]}v^+ = \pm f_{\alpha_{l-1} + \alpha_l}v^+ \in V_{l-1,l} = V_{r,l}$.

So the Lemma is proved. \diamond

Now let a_j be the first non-zero label after a_i . Since there are fewer non-zero labels between a_i and a_l than between a_i and a_m , by our inductive hypothesis $f_{[i,j]}v^+ \in V_{i,j}$. So the Proposition is proved. \diamond

Now the Proposition and Lemma 4 tell us that $f_{[i,j]}v^+ \in V_{i,j}$ for any pair of successive non-zero labels a_i and a_j . The following Lemma completes the proof of the theorem.

LEMMA 7 *Let $1 \leq i < j \leq n$ such that $a_i \neq 0 \neq a_j, a_l = 0$ for $i < l < j$. Then $f_{[i,j]}v^+ \in V_{i,j}$ if and only if $a_i + a_j \equiv i - j \pmod{p}$.*

PROOF: Under the hypotheses, the $\lambda - (\alpha_i + \dots + \alpha_j)$ -weight space is spanned by $\{F_l v^+ = f_{[i,l]}f_{[l+1,j]}v^+ | i \leq l < j\} \cup \{f_{[i,j]}v^+\}$. So $f_{[i,j]}v^+ \in V_{i,j}$ if and only if there is a relation

$$0 = f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+. \quad (**)$$

The vector on the right hand side of (**) is 0 if and only if it is killed by all e_β 's (as it is not a high weight vector):

$$\begin{aligned} 0 &= e_{\alpha_j} \left(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+ \right) = \\ &\quad f_{[i,j-1]}v^+ + a_j b_{j-1} f_{[i,j-1]}v^+ \quad \Leftrightarrow a_j b_{j-1} = -1 \\ 0 &= e_{\alpha_{j-1}} \left(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+ \right) = \\ &\quad -b_{j-2} f_{[i,j-2]} f_{\alpha_j} v^+ + b_{j-1} f_{[i,j-2]} f_{\alpha_j} v^+ \quad \Leftrightarrow b_{j-2} = b_{j-1} \\ 0 &= e_{\alpha_{j-2}} \left(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+ \right) = \\ &\quad -b_{j-3} f_{[i,j-3]} f_{\alpha_{j-1} + \alpha_j} v^+ + b_{j-2} f_{[i,j-3]} f_{\alpha_{j-1} + \alpha_j} v^+ \Leftrightarrow b_{j-3} = b_{j-2} \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
0 &= e_{\alpha_{i+1}} (f_{[i,j]} v^+ + \sum_{l=i}^{j-1} b_l F_l v^+) = \\
&\quad - b_i f_{\alpha_i} f_{[i+2,j]} v^+ + b_{i+1} f_{\alpha_i} f_{[i+2,j]} v^+ \quad \Leftrightarrow b_i = b_{i+1} \\
0 &= e_{\alpha_i} (f_{[i,j]} v^+ + \sum_{l=i}^{j-1} b_l F_l v^+) = -f_{[i+1,j]} v^+ + (a_i + 1) b_i f_{[i+1,j]} v^+ \\
&\quad + b_{i+1} f_{[i+1,j]} v^+ + \cdots + b_{j-1} f_{[i+1,j]} v^+ \\
&\quad \Leftrightarrow b_i (a_i + 1) + \sum_{l=i+1}^{j-1} b_l = 1
\end{aligned}$$

(This uses the fact that we may choose the sign of $N(\alpha, \beta)$ for all extraspecial pairs α, β (see [0, p. 58]); we choose them so that $[e_{\alpha_{i+1} + \cdots + \alpha_j}, e_{\alpha_i}] = -e_{\alpha_i + \cdots + \alpha_j}$ for $l < j$, which forces all of the above signs.)

For $m < i$ or $m > j$, $e_{\alpha_m} (f_{\alpha_i + \cdots + \alpha_j} v^+ + \sum_{l=i}^{j-1} b_l F_l v^+) = 0$ trivially.

This gives a series of equations $b_i = \cdots = b_{j-1} = \frac{-1}{a_j}$. The last equation (for e_{α_i}) gives $\frac{1}{a_j} (a_i + 1) + \frac{1}{a_j} (j - i - 1) = -1$, or $a_j + a_i = i - j$. \diamond

So the Theorem is proved, and the conjecture of Jantzen and Seitz follows. \diamond

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In this paper Sym_n denotes the symmetric group on n objects, and Sym_{n-1} is embedded in Sym_n as the stabilizer of one of the objects. Representations are understood to be over an algebraically closed field K of characteristic $p > 0$. The methods used here were originally developed in work on the restriction of representations of algebraic groups; see [2] and [3].

We recall a few facts from the representation theory of the symmetric groups (see [5] for more). The irreducible $K\text{Sym}_n$ -modules are parameterized by p -regular partitions $\mu = (\mu_1^{m_1}, \dots, \mu_l^{m_l})$ of n , where p -regular means that $m_i \leq p - 1$ for every $1 \leq i \leq l$.

A fundamental result in the representation theory of the symmetric groups over a field of characteristic 0 is the branching theorem, which describes exactly what

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happens when an irreducible representation of Sym_n is restricted to Sym_{n-1} . No similar result is known for representations in finite characteristic $p < n$, and finding branching theorems over fields of arbitrary characteristic was the object of [6]. A more limited problem is determining those representations of $K\text{Sym}_n$ which remain irreducible for $K\text{Sym}_{n-1}$. In [6], Jantzen and Seitz state the following conjecture, proving the “if” direction (for $p = 2$ the conjecture was made by D. Benson):

CONJECTURE 1 *Let M be an irreducible $K\text{Sym}_n$ -module corresponding to the p -regular partition $\mu = (\mu_1^{m_1}, \dots, \mu_l^{m_l})$. Then $M|_{\text{Sym}_{n-1}}$ is irreducible if and only if $\mu_i - \mu_{i+1} + m_i + m_{i+1} \equiv 0 \pmod{p}$, for $i = 1, \dots, l - 1$.*

Note: Kleshchev gives a different proof of this conjecture in [7]. His proof consists of two pieces; the first, his Corollary 2.11, is quoted below (Proposition 3) and is the easier of the two. The second part (Theorem 7.2 in [7], showing — in the notation used following Proposition 3 below — that if V_1 is an irreducible L' -module, then the congruences on the coefficients of λ hold) requires a long computational proof, with considerable attention paid to the exact values of many constants. As the proof presented here is considerably shorter and, we believe, more transparent, it seemed worthwhile to present it.

This work was mentioned in the original paper of Jantzen and Seitz ([6]), which motivated Kleshchev’s work, as going “a long way towards establishing the conjecture.” In fact, all that was missing was Corollary 2.11 of [7].

One well-known connection between the representation theory of the symmetric groups and that of the general linear groups is the Schur functor: Starting with a representation of $GL_n(K)$, one gets an action of Sym_n on the weight space corresponding to the determinant character. Thus, a map from rational representations of $GL_n(K)$ to K -representations of Sym_n is obtained. Jantzen and Seitz exploit this correspondence to prove that Conjecture 1 is implied by Theorem 2 below.

Let $Y = SL_n(K)$, and let $\{\alpha_1, \dots, \alpha_{n-1}\} = \Pi(Y)$ be the set of fundamental roots of the root system $\Sigma(Y) = \Sigma$ (with respect to a fixed Borel subgroup B

and maximal torus T). The fundamental dominant weights of T will be denoted by $\lambda_1, \dots, \lambda_{n-1}$, with $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n-1$. Let $V = V(\lambda)$ be a non-trivial irreducible restricted high weight module for Y of highest weight $\lambda = a_1\lambda_1 + a_2\lambda_2 + \dots + a_{n-1}\lambda_{n-1}$, and let a_{i_1}, \dots, a_{i_m} be the non-zero a_i , with $i_1 < i_2 < \dots < i_m = k$. Finally, P will be the maximal parabolic subgroup of Y corresponding to $\alpha_{n-1} \in \Pi(Y)$ and $P = QL$ its Levi decomposition (with respect to T), so $L' \cong SL_{n-1}$. If μ is a weight of T , then V_μ is the weight space in V of weight μ .

Let $V_1 = \bigoplus V_\mu$, with the sum taken over those $\mu = \lambda - (b_1\alpha_1 + b_2\alpha_2 + \dots + b_{n-1}\alpha_{n-1})$ with $b_{n-1} = 1$. Let v^+ be a non-zero T -high weight vector in V .

We now have enough notation to state the main result, which implies Conjecture 1:

THEOREM 2 *V_1 has exactly one restricted L' -composition factor if and only if*

$$a_{i_j} + a_{i_{j+1}} + i_{j+1} - i_j \equiv 0 \pmod{p}$$

for all $j = 1, \dots, m-1$.

PROOF: The ‘‘if’’ direction of this was proved in [6]. In [7], Kleshchev proves

PROPOSITION 3 *V_1 has exactly one restricted L' -composition factor if and only if V_1 is an irreducible L' -module.*

So our task is to show that if V_1 is an irreducible L' -module, then the congruences on the coefficients of λ hold.

Since we are assuming that V is a restricted Y -module, we may work over the Lie algebra instead of the group. We use the standard basis $\{e_\alpha, f_\alpha, h_i | \alpha \in \Sigma^+(Y), 1 \leq i \leq n-1\}$ for the Lie algebra of Y , and we let $V_{i,l}$ be the subspace of $V_{\lambda - (\alpha_i + \dots + \alpha_l)}$ spanned by all elements other than $f_{\alpha_i + \dots + \alpha_l} v^+$ of the form

$$f_{\alpha_i + \dots + \alpha_j} f_{\alpha_{j+1} + \dots + \alpha_{j'}} \cdots f_{\alpha_{j(m)+1} + \dots + \alpha_l} v^+, \quad (\dagger)$$

with $i < j < \dots < j^{(m)} < l$. To ease the notational pain, we write $f_{[i,j]}$ for $f_{\alpha_i + \dots + \alpha_j}$.

The L' -module V_1 has a filtration

$$\begin{aligned} 0 &\leq \langle f_{[k,n-1]}v^+ \rangle \leq \langle f_{[k,n-1]}v^+, f_{[k-1,n-1]}v^+ \rangle \\ &\leq \langle f_{[k,n-1]}v^+, f_{[k-1,n-1]}v^+, f_{[k-2,n-1]}v^+ \rangle \\ &\leq \dots \leq V_1, \end{aligned}$$

where a_k is the “last” non-zero coefficient of λ , i.e. $a_l = 0$ for every $l > k$. Since V_1 is irreducible as an L' -module, this filtration has only one member, i.e. $f_{[i,n-1]}v^+ \in \langle f_{[k,n-1]}v^+ \rangle$ for all $i < k$. In particular, $f_{[i,n-1]}v^+ \in V_{i,n-1}$ for all $i < k$.

LEMMA 4 *Let a_k be the last non-zero coefficient of λ . If $i < k$, then $f_{[i,k]}v^+ \in V_{i,k}$.*

PROOF: To say that $f_{[i,n-1]}v^+ \in V_{i,n-1}$ is to say that there is a relation

$$f_{[i,n-1]}v^+ = \sum a_{(j,j',\dots)} f_{[i,j]} f_{[j+1,j']} \cdots f_{[j^{(m)},n-1]}v^+, \quad (*)$$

for some constants $a_{(j,j',\dots)}$, with $m \geq 2$ for every summand and $i < j < j' < \dots < j^{(m)} \leq k$. Notice that for any $l < k$, $f_{[l,n-1]}v^+ = a f_{[k+1,n-1]} f_{[l,k]}v^+$ for some non-zero $a \in K$, as $f_{[k+1,n-1]}v^+ = 0$. Similarly, for any $l > k$, $f_{[l,n-1]}v^+ = 0$. So (*) may be rewritten as

$$f_{[k+1,n-1]} f_{[i,k]}v^+ = f_{[k+1,n-1]} \sum b_{(j,j',\dots)} f_{[i,j]} \cdots f_{[j^{(m)},k]}v^+$$

for some constants $b_{(j,j',\dots)}$. We apply $e_{\alpha_{k+1} + \dots + \alpha_{n-1}}$ and see that this in turn implies

$$f_{[i,k]}v^+ = \sum c_{(j,j',\dots)} f_{[i,j]} \cdots f_{[j^{(m)},n-1]}v^+$$

for some constants $c_{(j,j',\dots)}$, since $(\lambda - (\alpha_i + \dots + \alpha_k))(h_{\alpha_{k+1} + \dots + \alpha_{n-1}}) \neq 0$. \diamond

PROPOSITION 5 *Let V and λ be as above. Let a_i, a_m be non-zero coefficients with $m > i$. If $f_{[r,m]}v^+ \in V_{r,m}$ for all r with $i \leq r < m$, then $f_{[i,j]}v^+ \in V_{i,j}$, where a_j is the first non-zero label after a_i .*

PROOF: Assume the hypotheses. If $j = m$, the Proposition is vacuous; so assume there are non-zero coefficients between a_i and a_m ; we proceed by induction on the number of such non-zero coefficients. Let a_l be the last non-zero coefficient before a_m .

For the calculations below, recall that since v^+ is a maximal vector, $e_\beta v^+ = 0$ for any $\beta \in \Sigma^+$, and that e_β and f_δ commute whenever $\beta - \delta \notin \Sigma$. If $\beta - \delta \in \Sigma$, then $e_\beta f_\delta = f_\delta e_\beta + d e_{\beta-\delta}$ for some $d = N(\beta, -\delta) \in \mathbb{Z}$ (where $e_{-\alpha} = f_\alpha$ for $\alpha \in \Sigma^+$). In our case ($Y = SL_n$), $d = \pm 1$.

We require the following lemma to complete the proof of the Proposition.

LEMMA 6 *If a_i , a_l , and a_m are non-zero coefficients of λ with $i < l < m$ and a_l is the last non-zero coefficient before a_m , then $f_{[r,m]}v^+ \in V_{r,m}$ for all r with $i \leq r < m$ implies that $f_{[r,l]}v^+ \in V_{r,l}$ for all r with $i \leq r < l$.*

PROOF: Assume that there is an $r \in [i, l-1]$ such that $f_{[r,l]}v^+ \notin V_{r,l}$.

The assumption that $f_{[r,m]}v^+ \in V_{r,m}$ implies that there is some non-zero linear combination

$$0 = f_{[r,m]}v^+ + \sum_{(\rho_1, \dots, \rho_s)} d_{(\rho_1, \dots, \rho_s)} f_{\rho_1} \cdots f_{\rho_s} v^+, \quad (*)$$

where $f_{\rho_1} \cdots f_{\rho_s} v^+$ is a term of type (\dagger) , with $s \geq 2$ and $\sum \rho_j = \alpha_r + \cdots + \alpha_m$.

Now we apply some $e_{\gamma_1} \cdots e_{\gamma_q}$ to $(*)$, with $\gamma_1 = \alpha_{l+1} + \cdots + \alpha_l$, $\gamma_2 = \alpha_{l+1} + \cdots + \alpha_{l-1}$, ..., and $\sum \gamma_j = \alpha_{l+1} + \cdots + \alpha_m$. Notice that such a product of e_γ 's applied to a summand of $(*)$ gives a multiple of a generator for $V_{\lambda - (\alpha_r + \cdots + \alpha_l)}$ of type (\dagger) . Consider those summands which give a multiple of $f_{[r,l]}v^+$. Since each f_{ρ_a} must involve some α_q with $a_q \neq 0$, these terms are exactly the

$$F_j v^+ = f_{[r, l+j]} f_{[l+j+1, m]} v^+, \quad 0 \leq j \leq m-l,$$

since all the coefficients between a_l and a_m are 0. Notice that $j = m-l$ gives $F_{m-l} v^+ = f_{[r,m]} v^+$.

Let the coefficient of $F_j v^+$ in $(*)$ be b_j ($= d_{(\alpha_r + \cdots + \alpha_{l+j}, \alpha_{l+j+1} + \cdots + \alpha_m)}$); note that $b_{m-l} = 1$. Since by our assumption $f_{[r,l]}v^+$ is not a combination of other terms

(†) which appear in the $\lambda - (\alpha_r + \cdots + \alpha_l)$ -weight space, any $e_{\gamma_1} \cdots e_{\gamma_q}$ (with the γ_a 's as above) must kill $\sum b_j F_j v^+$. In particular,

$$E_s = e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} e_{\alpha_{l+s+1} + \cdots + \alpha_m}, \quad 0 \leq s < m - l,$$

must kill this sum.

Now assume $f_{[r, l-1]} v^+ \notin V_{r, l-1}$. Consider the summands in (*) which give a multiple of $f_{[r, l-1]} v^+$ when some $e_{\gamma_1} \cdots e_{\gamma_q}$ (where $\gamma_j \succ 0$ and $\sum \gamma_j = \alpha_l + \cdots + \alpha_m$) is applied. They are the $F_j v^+$ ($0 \leq j \leq m - l$), and

$$G_j v^+ = f_{[r, l-1]} f_{[l, l+j]} f_{[l+j+1, m]} v^+, \quad 0 \leq j < m - l$$

(note that we need not include $G_{m-l} = f_{[r, l-1]} f_{[l, m]} v^+$ here by our assumption that $f_{[l, m]} v^+$ is a consequence of the terms already listed).

By the assumption that $f_{[r, l-1]} v^+ \notin V_{r, l-1}$, it follows that $f_{[r, l-1]} v^+ \neq 0$. So any $e_{\gamma_1} \cdots e_{\gamma_q}$ (as above) must kill

$$\sum_{j=0}^{m-l} b_j F_j v^+ + \sum_{j=0}^{m-l-1} c_j G_j v^+,$$

where c_j is the coefficient of G_j in (*). In particular, $e_{\alpha_l} E_s$ must kill $\sum c_j G_j v^+$ for every $0 \leq s < m - l$. The following argument shows that $E_s G_j v^+ = 0$ if $s \neq j$: Assume $s < j$. Then

$$\begin{aligned} E_s G_j v^+ &= (e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} e_{\alpha_{l+s+1} + \cdots + \alpha_m} f_{[r, l-1]} f_{[l, l+j]} f_{[l+j+1, m]}) v^+ \\ &= e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} (f_{[r, l-1]} f_{[l, l+j]} e_{\alpha_{l+s+1} + \cdots + \alpha_m} f_{[l+j+1, m]} v^+) \\ &= \pm e_{\alpha_{l+1} + \cdots + \alpha_{l+s}} (f_{[r, l-1]} f_{[l, l+j]} e_{\alpha_{l+s+1} + \cdots + \alpha_{l+j}} v^+) = 0. \end{aligned}$$

A similar calculation holds for $s > j$. Also,

$$\begin{aligned} e_{\alpha_l} E_j G_j v^+ &= e_{\alpha_l} f_{[r, l-1]} e_{\alpha_{l+1} + \cdots + \alpha_{l+j}} f_{[l, l+j]} e_{\alpha_{l+j+1} + \cdots + \alpha_m} f_{[l+j+1, m]} v^+ \\ &= \pm e_{\alpha_l} f_{[r, l-1]} f_{\alpha_l} h_{\alpha_{l+j+1} + \cdots + \alpha_m} v^+ \\ &= \pm a_l a_m f_{[r, l-1]} v^+. \end{aligned}$$

So:

$$0 = e_{\alpha_l} E_0(\sum c_j G_j v^+) = \pm c_0 a_l a_m f_{[r, l-1]} v^+ \Rightarrow c_0 = 0$$

$$0 = e_{\alpha_l} E_1(\sum c_j G_j v^+) = \pm c_1 a_l a_m f_{[r, l-1]} v^+ \Rightarrow c_1 = 0$$

⋮

$$0 = e_{\alpha_l} E_{m-l-1}(\sum c_j G_j v^+) = \pm c_{m-l-1} a_l a_m f_{[r, l-1]} v^+ \Rightarrow c_{m-l-1} = 0$$

So this implies $\sum c_j G_j v^+ = 0$. But then:

$$\begin{aligned} 0 &= e_{\alpha_l + \dots + \alpha_m} (\sum b_j F_j v^+ + \sum c_j G_j v^+) = e_{\alpha_l + \dots + \alpha_m} (\sum b_j F_j v^+) \\ &= \pm f_{[r, l-1]} v^+ \end{aligned}$$

(as $F_{m-l} v^+ = f_{[r, m]} v^+$ is the only term in $\sum b_j F_j v^+$ not killed by $e_{\alpha_l + \dots + \alpha_m}$). But this is a contradiction.

So our assumption that there was an r , $i \leq r < l$, such that $f_{[r, l]} v^+ \notin V_{r, l}$ and $f_{[r, l-1]} v^+ \notin V_{r, l-1}$ must be false; i.e. for every r , $i \leq r < l$, either (i) $f_{[r, l]} v^+ \in V_{r, l}$, or (ii) $f_{[r, l-1]} v^+ \in V_{r, l-1}$. We want to show that in fact (i) holds always. There are several cases:

(a) If $r < l - 1$ and $f_{[r, l-1]} v^+ \in V_{r, l-1}$, we see that:

$$\begin{aligned} f_{[r, l]} v^+ &= f_{\alpha_l} f_{[r, l-1]} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \\ &= \sum_{(\rho_1, \dots, \rho_j), j \geq 2} f_{\alpha_l} f_{\rho_1} \cdots f_{\rho_j} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \\ &= \sum f_{\rho_1} \cdots f_{\rho_{j-1}} f_{\alpha_l} f_{\rho_j} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \\ &= \sum f_{\rho_1} \cdots f_{\rho_j} f_{\alpha_l} v^+ + s f_{\rho_1} \cdots f_{\rho_j + \alpha_l} v^+ - f_{[r, l-1]} f_{\alpha_l} v^+ \end{aligned}$$

for some integers s (depending on (ρ_1, \dots, ρ_j)), so $f_{[r, l]} v^+ \in V_{r, l}$.

(b) If $r = l - 1$ and $a_{l-1} \neq 0$, (ii) cannot happen, since $V_{l-1, l-1} = 0$ and $f_{[r, l-1]} v^+ = f_{\alpha_{l-1}} v^+ \neq 0$.

(c) If $r = l - 1$ and $a_{l-1} = 0$, then

$$f_{\alpha_{l-1} + \alpha_l} v^+ = \pm (f_{\alpha_l} f_{\alpha_{l-1}} v^+ - f_{\alpha_{l-1}} f_{\alpha_l} v^+) = \pm f_{\alpha_{l-1}} f_{\alpha_l} v^+,$$

since $a_{l-1} = 0$. So $f_{[r,l]}v^+ = \pm f_{\alpha_{l-1} + \alpha_l}v^+ \in V_{l-1,l} = V_{r,l}$.

So the Lemma is proved. \diamond

Now let a_j be the first non-zero label after a_i . Since there are fewer non-zero labels between a_i and a_l than between a_i and a_m , by our inductive hypothesis $f_{[i,j]}v^+ \in V_{i,j}$. So the Proposition is proved. \diamond

Now the Proposition and Lemma 4 tell us that $f_{[i,j]}v^+ \in V_{i,j}$ for any pair of successive non-zero labels a_i and a_j . The following Lemma completes the proof of the theorem.

LEMMA 7 *Let $1 \leq i < j \leq n$ such that $a_i \neq 0 \neq a_j, a_l = 0$ for $i < l < j$. Then $f_{[i,j]}v^+ \in V_{i,j}$ if and only if $a_i + a_j \equiv i - j \pmod{p}$.*

PROOF: Under the hypotheses, the $\lambda - (\alpha_i + \dots + \alpha_j)$ -weight space is spanned by $\{F_l v^+ = f_{[i,l]}f_{[l+1,j]}v^+ | i \leq l < j\} \cup \{f_{[i,j]}v^+\}$. So $f_{[i,j]}v^+ \in V_{i,j}$ if and only if there is a relation

$$0 = f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+. \quad (**)$$

The vector on the right hand side of (**) is 0 if and only if it is killed by all e_β 's (as it is not a high weight vector):

$$\begin{aligned} 0 &= e_{\alpha_j} \left(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+ \right) = \\ &\quad f_{[i,j-1]}v^+ + a_j b_{j-1} f_{[i,j-1]}v^+ \quad \Leftrightarrow a_j b_{j-1} = -1 \\ 0 &= e_{\alpha_{j-1}} \left(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+ \right) = \\ &\quad -b_{j-2} f_{[i,j-2]} f_{\alpha_j} v^+ + b_{j-1} f_{[i,j-2]} f_{\alpha_j} v^+ \quad \Leftrightarrow b_{j-2} = b_{j-1} \\ 0 &= e_{\alpha_{j-2}} \left(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+ \right) = \\ &\quad -b_{j-3} f_{[i,j-3]} f_{\alpha_{j-1} + \alpha_j} v^+ + b_{j-2} f_{[i,j-3]} f_{\alpha_{j-1} + \alpha_j} v^+ \Leftrightarrow b_{j-3} = b_{j-2} \\ &\quad \vdots \end{aligned}$$

$$\begin{aligned}
0 = e_{\alpha_{i+1}}(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+) &= \\
&- b_i f_{\alpha_i} f_{[i+2,j]}v^+ + b_{i+1} f_{\alpha_i} f_{[i+2,j]}v^+ \quad \Leftrightarrow b_i = b_{i+1} \\
0 = e_{\alpha_i}(f_{[i,j]}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+) &= -f_{[i+1,j]}v^+ + (a_i + 1)b_i f_{[i+1,j]}v^+ \\
&+ b_{i+1} f_{[i+1,j]}v^+ + \cdots + b_{j-1} f_{[i+1,j]}v^+ \\
&\Leftrightarrow b_i(a_i + 1) + \sum_{l=i+1}^{j-1} b_l = 1
\end{aligned}$$

(This uses the fact that we may choose the sign of $N(\alpha, \beta)$ for all extraspecial pairs α, β (see [1, p. 58]); we choose them so that $[e_{\alpha_{i+1}+\cdots+\alpha_j}, e_{\alpha_i}] = -e_{\alpha_i+\cdots+\alpha_j}$ for $l < j$, which forces all of the above signs.)

For $m < i$ or $m > j$, $e_{\alpha_m}(f_{\alpha_i+\cdots+\alpha_j}v^+ + \sum_{l=i}^{j-1} b_l F_l v^+) = 0$ trivially.

This gives a series of equations $b_i = \cdots = b_{j-1} = \frac{-1}{a_j}$. The last equation (for e_{α_i}) gives $\frac{1}{a_j}(a_i + 1) + \frac{1}{a_j}(j - i - 1) = -1$, or $a_j + a_i = i - j$. \diamond

So the Theorem is proved, and the conjecture of Jantzen and Seitz follows. \diamond

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