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# Self-conjugate simultaneous $p$ - and $q$ -core partitions and blocks of $A_n$

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## ABSTRACT

Given two relatively prime integers  $s$  and  $t$ , we prove that there are exactly  $\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}$  self-conjugate partitions that are simultaneously  $s$ -core and  $t$ -core. For distinct primes  $p$  and  $q$ , this leads to a count of  $p$ - and  $q$ -blocks for the alternating groups whose sets of ordinary irreducible characters coincide.

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## 1. Introduction

In [8], Navarro and Willems began an investigation into the relationship between  $p$ -blocks and  $q$ -blocks (for distinct primes  $p$  and  $q$ ) of representations of a finite group  $G$ . They conjectured that if the sets of irreducible ordinary characters in the  $p$ -block  $B_p$  and the  $q$ -block  $B_q$  coincide, then this set must consist of a single character – that is, the block has  $p$ -defect and  $q$ -defect 0 (see [4, Chapter 7] for block theory background). They proved the conjecture for several classes of groups. Below, when we say that a  $p$ -block and a  $q$ -block coincide, we mean the sets of ordinary irreducible characters in the blocks are the same.

Partitions of  $n$  parametrize the ordinary irreducible representations of the symmetric group  $S_n$ , and the so-called “Nakayama conjecture” gives a beautiful description of the classification of ordinary

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characters of  $S_n$  into  $p$ -blocks. Navarro and Willems' question has been pursued for the symmetric group: First, J. Anderson [1] showed that the number of defect-0  $p$ -blocks of symmetric groups that coincide with a  $q$ -block is  $\frac{\binom{p+q}{q}}{p+q}$ . More recently, J.B. Olsson and D. Stanton [10] note a counterexample to Navarro and Willems' general conjecture which was discovered by Bessenrodt, prove the conjecture for  $S_n$ , and show that the maximum  $n$  for which  $S_n$  has coinciding  $p$ - and  $q$ -blocks is  $n = \frac{(p^2-1)(q^2-1)}{24}$ , a result conjectured by Ben Kane in his Master's thesis [7]. In a subsequent paper [3], Bessenrodt, Navarro, Olsson and Tiep reduce the study of Navarro and Willems' original question – in the case of principal blocks – to the study of simple groups, and then confirm the conjecture in the case of principal blocks for all simple groups.

The irreducible representations of the alternating group  $A_n$  in odd characteristic can be obtained from those for the symmetric group via a simple application of Clifford theory. The irreducible representations of  $A_n$  in odd characteristic were indexed by partitions in this manner in [5]. From [9, Proposition 12.2], we know that every  $p$ -block of  $S_n$  restricts to  $A_n$  as a single  $p$ -block, except perhaps for those of defect 0 (that is, those corresponding to  $p$ -core partitions of  $n$  and containing a single irreducible ordinary representation); and of these, the Clifford theory argument tells us that those  $p$ -core partitions which are self-conjugate are precisely those whose corresponding  $p$ -block (of  $S_n$ ) splits on restriction to  $A_n$ .

Thus we can answer Navarro and Willems' original question for  $A_n$  ("When is a  $p$ -block a  $q$ -block?") if we can answer it for  $S_n$  and determine which coinciding  $p$ - and  $q$ -blocks for  $S_n$  split on restriction to  $A_n$ .

For  $S_n$ , the papers of Anderson [1] and Olsson and Stanton [10] provide the following answers: Given fixed primes  $p$  and  $q$ , Anderson's method for counting simultaneous  $p$ - and  $q$ -core partitions is constructive; and Olsson and Stanton show that these simultaneous  $p$ - and  $q$ -core partitions give all examples for  $S_n$  of  $p$ - and  $q$ -blocks that coincide. Finally, Olsson and Stanton show that if  $n > \frac{(p^2-1)(q^2-1)}{24}$ , then there is no simultaneous  $p$ - and  $q$ -core partition of  $n$  (and thus no  $p$ - and  $q$ -blocks of  $S_n$  that coincide).

In this paper, we develop a method to count the number of *self-conjugate* partitions which are simultaneously  $s$ - and  $t$ -core, for relatively prime positive integers  $s$  and  $t$ . Our main theorem gives this count, and its corollary gives the consequence for blocks of  $A_n$ .

**Theorem 1.** *Let  $s$  and  $t$  be relatively prime positive integers. Then there are*

$$\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}$$

*self-conjugate partitions which are both  $s$ -core and  $t$ -core.*

**Corollary 1.1.** *Let  $p$  and  $q$  be distinct odd primes.*

(1) *Among all  $p$ -blocks and  $q$ -blocks of representations of finite symmetric groups  $S_n$ , there are*

$$\binom{\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor}{\lfloor \frac{p}{2} \rfloor}$$

*coinciding  $p$ - and  $q$ -blocks which split on restriction to  $A_n$ . The largest  $n$  for which this occurs is  $n = \frac{(p^2-1)(q^2-1)}{24}$ .*

(2) *Among all  $p$ -blocks and  $q$ -blocks of representations of finite alternating groups  $A_n$ , there are*

$$\frac{3}{2} \binom{\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor}{\lfloor \frac{p}{2} \rfloor} + \frac{1}{2} \cdot \frac{\binom{p+q}{p}}{(p+q)}$$

*cases in which a  $p$ -block and a  $q$ -block coincide; the largest  $n$  for which this occurs is  $n = \frac{(p^2-1)(q^2-1)}{24}$ .*

**Proof.** The first part of the corollary follows from Theorem 1, together with the fact that a  $p$ -block of  $S_n$  of positive defect covers a unique  $p$ -block of  $A_n$  [9, Proposition 12.2] and Olsson and Stanton's result in [10] that the largest  $n$  which has a simultaneous  $p$ - and  $q$ -core partition is  $n = \frac{(p^2-1)(q^2-1)}{24}$  and Kane's result in [7] that there is a self-conjugate simultaneous  $p$ - and  $q$ -core partition of this size.

The non-self-conjugate pairs of blocks for  $S_n$  restrict to  $A_n$  as indecomposable modules, with conjugate pairs restricting isomorphically. The self-conjugate irreducible  $S_n$ -blocks restrict as a sum of two non-isomorphic  $A_n$ -blocks. As non-irreducible  $p$ -blocks and  $q$ -blocks of  $S_n$  do not coincide ([10] and [9, Proposition 12.2] again), the only possibility for  $p$ -blocks and  $q$ -blocks of  $A_n$  to coincide is when coinciding  $S_n$ -blocks restrict to  $A_n$ .

Thus the second part of the corollary follows from the first, together with Anderson's result in [1] that there are  $\frac{\binom{p+q}{p}}{p+q}$  simultaneous  $p$ - and  $q$ -core partitions.  $\square$

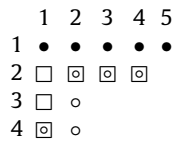
### 2. Preliminaries

A partition  $\Lambda = (\lambda_1, \dots, \lambda_d)$  of size  $|\Lambda| := \sum_{i=1}^d \lambda_i$  is any finite sequence of non-increasing positive integer parts  $\lambda_i$ . We will denote the number of partitions of an integer  $n$  by  $p(n)$ .

The Ferrers–Young diagram of a partition  $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$  is formed by arranging  $|\Lambda|$  nodes in rows so that the  $i$ th row has  $\lambda_i$  nodes. The nodes are labeled by row and column coordinates as one would label the entries of a matrix. The conjugate of  $\Lambda$ , denoted  $\Lambda^c$ , is the partition whose Ferrers–Young diagram is the reflection along the main diagonal of  $\Lambda$ 's diagram, and  $\Lambda$  is self-conjugate if  $\Lambda = \Lambda^c$ . A node's hook consists of the node along with any other nodes directly below or to the right of the node itself. The cardinality of a node's hook is the node's hook number. We will use  $H(i, j)$  to denote the hook number of the node at position  $(i, j)$ .

A node's rim hook is the sequence of connected nodes on the right-hand boundary of the Ferrers–Young diagram connecting the two end points of its hook. The two end nodes of the hook and rim hook are known as the hand (at the end of the node's row) and foot (at the bottom of the node's column) nodes. The rim hook and the hook of a node are necessarily of equal length; indeed, any path from the hand to the foot, consisting only of leftward and downward moves, will contain this same number of nodes. A  $t$ -core partition is a partition with no hook number divisible by  $t$ .

**Example 2.** The Ferrers–Young diagram for  $\Lambda = (5, 4, 2, 2)$  is below:



The hook number of the node at  $(2, 1)$  is  $H(2, 1) = 6$  because the hook (square nodes) and rim hook (open circle nodes) of  $(2, 1)$  each consists of six nodes. The hand and foot nodes are  $(2, 4)$  and  $(4, 1)$ , respectively. Since the hook numbers are  $\{1, 2, 3, 4, 5, 6, 7, 8\}$ ,  $\Lambda$  is a  $t$ -core for  $t \geq 9$ .

Let

$$C(t) = \{\Lambda: \Lambda \text{ is a } t\text{-core partition}\},$$

$$SC(t) = \{\Lambda: \Lambda \text{ is a self-conjugate } t\text{-core partition}\},$$

$$C(t, n) = \{\Lambda: \Lambda \text{ is a } t\text{-core partition, } |\Lambda| = n\},$$

$$SC(t, n) = \{\Lambda: \Lambda \text{ is a self-conjugate } t\text{-core partition, } |\Lambda| = n\}.$$

Granville and Ono [6] proved that for  $t \geq 4$ , every natural number  $n$  has a  $t$ -core partition (i.e.  $|C(t, n)| > 0$  for  $t \geq 4, n > 0$ ). This settled a conjecture of Brauer regarding the existence of defect zero characters for finite simple groups. More recently, several authors [2] showed that a self-dual  $t$ -core partition exists for every  $n > 2$  provided that  $t = 8$  or  $t \geq 10$ .

### 3. Main diagonal hooks

We develop results for the main diagonal hooks of self-conjugate partitions. Throughout this section,  $\Lambda$  is a self-conjugate partition. We define the set of main diagonal hook numbers of  $\Lambda$  to be

$$MD(\Lambda) = \left\{ h \mid \begin{array}{l} h \text{ is the hook number of a node on the main diagonal} \\ \text{of the Ferrers–Young diagram of } \Lambda \end{array} \right\}.$$

It is clear that  $MD(\Lambda)$  determines  $\Lambda$  (under the assumption that  $\Lambda$  is self-conjugate).

We summarize the results of this section in

**Proposition 3.** *Let  $\Lambda$  be a self-conjugate partition. Then  $\Lambda$  is a  $t$ -core partition if and only if both of the following hold:*

- (1) *If  $h \in MD(\Lambda)$  with  $h > 2t$ , then  $h - 2t \in MD(\Lambda)$ ; and*
- (2) *If  $h, s \in MD(\Lambda)$ , then  $h + s \not\equiv 0 \pmod{2t}$ .*

Below, we need several lemmas which restrict the possible main diagonal hook numbers. The first is clear from the fact that conjugation interchanges the nodes below a main diagonal node with those to the right of the same main diagonal node.

**Lemma 4.** *If  $\Lambda$  is self-conjugate, then the entries of  $MD(\Lambda)$  are all odd.*

Now let  $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$ , and  $MD(\Lambda) = \{h_1, h_2, \dots, h_k\}$  with  $h_i > h_{i+1}$  for each  $i$ .

**Lemma 5.** *If  $\Lambda \in SC(t)$  and  $h_i \in MD(\Lambda)$  with  $h_i > 2t$ , then  $h_i - 2t \in MD(\Lambda)$ .*

**Proof.** Assume  $\Lambda \in SC(t)$  and  $h_i \in MD(\Lambda)$  with  $h_i > 2t$  and  $h_i - 2t \notin MD(\Lambda)$ . Choose  $j$  such that  $h_j > h_i - 2t, h_{j+1} < h_i - 2t$  (with  $j = k$  if there is no such  $j$ ; in this case, set  $h_{j+1} = -1$ ).

We consider the hooks of the nodes at positions  $(i, j + 1 + \frac{h_{j+1}+1}{2})$  through  $(i, j + \frac{h_j-1}{2})$ ; these are exactly the nodes in row  $i$  whose feet lie in row  $j$ . Since their feet are all in the same row, their hook lengths differ by 1 as we move along the row; that is, every hook length between  $H(i, j + 1 + \frac{h_{j+1}+1}{2})$  and  $H(i, j + \frac{h_j-1}{2})$  occurs in  $\Lambda$ . However, a simple computation shows that the restrictions on  $h_j$  and  $h_{j+1}$  ensure that  $t$  occurs in this list, contradicting the assumption that  $\Lambda \in SC(t)$ .  $\square$

The second restriction on allowable main diagonal hook numbers appears in this lemma.

**Lemma 6.** *If  $\Lambda \in SC(t)$  and  $h_i, h_j \in MD(\Lambda)$ , then  $h_i + h_j \not\equiv 0 \pmod{2t}$ .*

**Proof.**  $H(i, j) = \frac{h_i+h_j}{2}$  for  $i, j \leq k$ . Thus, if  $h_i + h_j \equiv 0 \pmod{2t}$  then  $\Lambda$  is not a  $t$ -core partition.  $\square$

Together, the two lemmas above establish the necessity of the two conditions in Proposition 3.

To establish sufficiency, assume  $\Lambda$  is not a  $t$ -core partition. Then  $t \mid H(i, j)$  for some node  $(i, j)$ ; say  $H(i, j) = ct$ . If  $i, j \leq k$  (recall  $MD(\Lambda) = \{h_1, h_2, \dots, h_k\}$ ), then  $H(i, j) = \frac{h_i+h_j}{2}$ , which implies  $h_i + h_j \equiv 0 \pmod{2t}$ , violating condition (2) in Proposition 3.

If one of  $i$  or  $j$  is larger than  $k$ , then by self-conjugacy we can assume without loss of generality that  $j > k \geq i$  (it is not possible for both  $i > k$  and  $j > k$ , as then  $(i, j)$  would not be a node of  $\Lambda$ ). Let  $(s, j)$  be the foot of the hook of node  $(i, j)$ ; note that  $s \leq k$  since  $j > k$ .

We can create a path from the hand of the  $(i, i)$  hook to its foot from the union of the  $(i, j)$  and  $(j, i)$  hooks, together with a subset of the  $(s, s)$  hook. The nodes  $(s, j)$  and  $(j, s)$  appear in the  $(i, j)$  and  $(j, i)$  hooks, respectively, so we need use only a proper subset of the  $(s, s)$  hook. Thus  $h_s = H(s, s) > H(i, i) - H(i, j) - H(j, i) = h_i - 2ct$ .

If  $s = k$ , this violates condition (1) in Proposition 3 (applied  $c$  times). If  $s < k$ , then note that  $(s + 1, j) \notin \Lambda$  (as  $(s, j)$  is at the bottom of a column). This forces  $h_{j+1} < h_i - 2ct$ , reasoning as above. Again, this violates condition (1).

This establishes Proposition 3.

**4. Simultaneous cores**

We apply the results of the previous section to partitions which are both  $s$ - and  $t$ -core. The restrictions on allowable main diagonal hook numbers allow us to arrange them in an array; certain paths through this array correspond to the possible partitions. This leads to the count in Theorem 1.

Let  $\Lambda \in SC(t)$ . Note that the partition formed by removing the top row and left column of  $\Lambda$  (that is, removing the hook of the node  $(1, 1)$ ) is also in  $SC(t)$ . Thus we can view any self-conjugate  $t$ -core as built up from the successive addition of larger main diagonal hooks.

Of course, if  $h \in MD(\Lambda)$ , then  $h \not\equiv t \pmod{2t}$ , as  $\Lambda$  is a  $t$ -core partition (this also follows from Lemma 6). Thus, if we consider the set of residues modulo  $2t$  that appear in  $MD(\Lambda)$ , we find that the neither the even classes (Lemma 4) nor the class containing  $t$  can occur. This leaves  $2\lfloor \frac{t}{2} \rfloor$  possible residue classes, and Lemma 6 ensures that of each pair of residue classes  $\{i, 2t - i\}$ , at most one can occur. Finally, Lemma 5 tells us that  $MD(\Lambda)$  (and thus  $\Lambda$ ) is determined by the largest entry from each residue class (mod  $2t$ ) that occurs in  $MD(\Lambda)$ .

Now we let  $\Lambda$  be both  $s$ - and  $t$ -core, for  $s$  and  $t$  relatively prime and  $s < t$ . Then the previous paragraph applies with  $t$  replaced by  $s$ , and the resulting conditions on the entries in  $MD(\Lambda)$  give strong restrictions.

We build an  $\lfloor \frac{s}{2} \rfloor \times \lfloor \frac{t}{2} \rfloor$  array  $A$  of integers and show that the absolute values of the array entries are exactly the possible main diagonal hook numbers for  $\Lambda$ . For  $1 \leq i \leq \lfloor \frac{s}{2} \rfloor$  and  $1 \leq j \leq \lfloor \frac{t}{2} \rfloor$ , set

$$A_{i,j} = st - s(2j - 1) - t(2i - 1).$$

We give the array  $A$  for  $s = 11, t = 13$  as an example:

119	97	75	53	31	9
93	71	49	27	5	-17
67	45	23	1	-21	-43
41	19	-3	-25	-47	-69
15	-7	-29	-51	-73	-95

The  $\lfloor \frac{s}{2} \rfloor$  rows represent distinct congruence classes modulo  $2s$ , since if  $A_{i,j} \equiv A_{k,l} \pmod{2s}$  (for  $i \neq k$ ), then  $s|i - k$ , which is impossible for  $1 \leq i, k \leq \lfloor \frac{s}{2} \rfloor$ . The congruence class of  $s \pmod{2s}$  is not represented, as this would imply  $s|(s - 2i + 1)$ , which is impossible for  $1 \leq i \leq \lfloor \frac{s}{2} \rfloor$ . Furthermore, if  $A_{i,j} \equiv -A_{k,l} \pmod{2s}$ , then  $s|(i + k - 1)$ , which is similarly impossible; thus, the  $\lfloor \frac{s}{2} \rfloor$  classes represented by the rows are distinct from those represented by their opposites. Since all entries in the array are odd, we have that all  $\lfloor \frac{s}{2} \rfloor$  potential congruence classes modulo  $2s$  mentioned above are represented by the absolute values of the entries in the rows (with the rows perhaps extended to ensure they have both positive and negative entries). Each row represents one of the  $\lfloor \frac{s}{2} \rfloor$  pairs of residues of the form  $(i, 2s - i)$ .

A similar argument demonstrates that the columns (possibly extended) represent all possible congruence classes modulo  $2t$  which could occur in  $MD(\Lambda)$ .

The array is constructed so that if any positive entry  $m$  occurs as a main diagonal hook number of  $\Lambda$ , then all smaller positive entries in the same row or column as  $m$  must also occur by Lemma 5. If the absolute value  $m$  of a negative entry in  $A$  occurs, then the absolute values of all larger negative entries in the same row or column must also occur. Lemma 6 ensures that if a positive entry  $m$  from  $A$  occurs in  $MD(\Lambda)$ , then the absolute values of the negative entries in the same row or column as  $m$  do not occur in  $MD(\Lambda)$ ; and if the absolute value of a negative entry occurs, then the positive entries in the same row or column do not occur.

**Lemma 7.** *Let  $\Lambda$  be a self-conjugate  $s$ - and  $t$ -core partition, for  $s$  and  $t$  relatively prime. All possible main diagonal hook numbers of  $\Lambda$  are contained in the array  $A$  as constructed above (for  $1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq \lfloor \frac{t}{2} \rfloor$ ).*

**Proof.** As described above, every possible entry in  $MD(\Lambda)$  occurs as the absolute value of an entry in one of the extended rows of  $A$ , as each possible  $2s$ -congruence class for an  $MD(\Lambda)$ -entry is represented in one of the rows; similarly, each entry in  $MD(\Lambda)$  occurs as the absolute value of an entry in one of the extended columns.

Note that if we extend the array one step in each direction, then we can determine the sign of all the new entries (except the corners). Straightforward calculations show that  $-2t < A_{1, \lfloor \frac{t}{2} \rfloor} < 2s$ , so  $A_{0, \lfloor \frac{t}{2} \rfloor} > 0$  and  $A_{1, \lfloor \frac{t}{2} \rfloor + 1} < 0$ . Thus, the entries in the “extended array” in positions  $(0, 1)$  through  $(0, \lfloor \frac{t}{2} \rfloor)$  (the top edge) are positive, and those in positions  $(1, \lfloor \frac{t}{2} \rfloor + 1)$  through  $(\lfloor \frac{s}{2} \rfloor, \lfloor \frac{t}{2} \rfloor + 1)$  (the right edge) are negative. Similarly, those in positions  $(1, 0)$  through  $(\lfloor \frac{s}{2} \rfloor, 0)$  (left edge) are positive, and those in  $(\lfloor \frac{s}{2} \rfloor + 1, 1)$  through  $(\lfloor \frac{s}{2} \rfloor + 1, \lfloor \frac{t}{2} \rfloor)$  (bottom edge) are negative. We show this graphically:

$$\begin{array}{cccc}
 & + & \dots & \oplus \\
 + & A_{1,1} & \dots & A_{1, \lfloor \frac{t}{2} \rfloor} & \ominus \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 \oplus & A_{\lfloor \frac{s}{2} \rfloor, 1} & \dots & A_{\lfloor \frac{s}{2} \rfloor, \lfloor \frac{t}{2} \rfloor} & - \\
 \ominus & \dots & & - & 
 \end{array}$$

By construction, the occurrence in  $MD(\Lambda)$  of a positive entry from a column (row) forces the inclusion in  $MD(\Lambda)$  of every smaller entry in the same column (row) by Lemma 5; similarly for negative entries. Thus the inclusion in  $MD(\Lambda)$  of any row or column entry not in  $A$  forces the inclusion of one of the entries labeled  $\oplus$  or  $\ominus$  in the diagram above. So to prove the lemma, it suffices to prove that none of (the absolute value of) these four entries can occur in  $MD(\Lambda)$ .

We calculate the absolute value of each of these four entries:

$$\begin{aligned}
 A_{0, \lfloor \frac{t}{2} \rfloor} &= st - s \left( 2 \left\lfloor \frac{t}{2} \right\rfloor - 1 \right) - t + 2t = \begin{cases} s + t & \text{if } t \text{ is even,} \\ 2s + t & \text{if } t \text{ is odd,} \end{cases} \\
 A_{\lfloor \frac{s}{2} \rfloor, 0} &= st - s - t \left( 2 \left\lfloor \frac{s}{2} \right\rfloor - 1 \right) + 2s = \begin{cases} s + t & \text{if } s \text{ is even,} \\ s + 2t & \text{if } s \text{ is odd,} \end{cases} \\
 -A_{1, \lfloor \frac{t}{2} \rfloor + 1} &= - \left( st - s \left( 2 \left\lfloor \frac{t}{2} \right\rfloor - 1 \right) - t \right) = \begin{cases} s + t & \text{if } t \text{ is even,} \\ t & \text{if } t \text{ is odd,} \end{cases} \\
 -A_{\lfloor \frac{s}{2} \rfloor + 1, 1} &= - \left( st - s - t \left( 2 \left\lfloor \frac{s}{2} \right\rfloor - 1 \right) \right) = \begin{cases} s + t & \text{if } s \text{ is even,} \\ s & \text{if } s \text{ is odd.} \end{cases}
 \end{aligned}$$

Ruling out these potential main diagonal hook numbers is straightforward:  $s$  and  $t$  cannot occur because  $\Lambda$  is  $s$ - and  $t$ -core;  $2s + t$  and  $s + 2t$  cannot occur because Lemma 5 would then force  $t$  or  $s$  as main diagonal hook numbers; finally,  $s + t$  cannot occur because then Lemma 5 would force

$s + t - 2s = t - s \in MD(\Lambda)$ , and then Lemma 6 would be violated because  $(s + t) + (t - s) = 2t \equiv 0 \pmod{2t}$ .

So all potential main diagonal hook numbers for  $\Lambda$  occur in  $A$ .  $\square$

Given an entry  $A_{i,j}$ , we next wish to demonstrate that  $|A_{i,j}|$  is a main diagonal hook number of a self-dual  $s$ - and  $t$ -core partition. By Proposition 3, it will suffice to construct a set of integers containing  $|A_{i,j}|$  which satisfies the two conditions of that proposition.

If  $A_{i,j} > 0$ , let  $M$  be the set of all positive  $A_{k,l}$  with  $i \leq k$  and  $j \leq l$ . Then condition (1) of Proposition 3 is satisfied. Condition (2) can be violated only by the inclusion of positive and (absolute values of) negative entries from the same row or column, which does not occur in  $M$ . Thus setting  $MD(\Lambda) = M$  gives a self-dual  $s$ - and  $t$ -core partition  $\Lambda$  which has  $A_{i,j}$  as a main diagonal hook number. The construction for  $A_{i,m} < 0$  is similar: Set  $M = \{|A_{k,l}|: A_{k,l} < 0, k \leq i, l \leq j\}$ .

Each column (row) of  $A$  has a point above (to the left of) which the entries are positive, and below (to the right of) which are negative. We construct a grid around the array  $A$ , with the grid lines running between the entries and around the edges. The segments which divide rows and columns into positive and negative entries are highlighted. As an example, we show the grid for  $s = 11, t = 13$ :

119	97	75	53	31	9
93	71	49	27	5	-17
67	45	23	1	-21	-43
41	19	-3	-25	-47	-69
15	-7	-29	-51	-73	-95

Now we obtain a correspondence between paths through the above grid and sets of array entries in the following way: For any path  $p$  along the edges of the grid from the bottom left to the upper right, consisting only of movements up or right, we let  $M(p)$  be the set of absolute values of array entries which are trapped between  $p$  and the highlighted path above. Thus,

$$M(p) = \{|a|: a \in A \text{ and either (i) } a > 0 \text{ and } a \text{ is below } p, \text{ or (ii) } a < 0 \text{ and } a \text{ is above } p\}.$$

Note that the highlighted path itself corresponds to the empty set.

Finally, we can state the result which allows us to count the self-dual  $s$ - and  $t$ -core partitions.

**Proposition 8.** *If  $M$  is a set of entries of  $A$  that satisfy the conditions of Proposition 3 for both  $s$  and  $t$ , then  $M = M(p)$  for some path  $p$  as above, and for every such path  $p$ ,  $M(p)$  satisfies the conditions of Proposition 3 for both  $s$  and  $t$ .*

**Proof.** If  $p$  is a path as above, then for  $M(p)$ , both conditions for  $s$  follow immediately from the construction of the rows of the array; similarly, both conditions for  $t$  follow from the construction of the columns.

If  $M$  is a set of entries of  $A$  that satisfy the conditions of Proposition 3 for both  $s$  and  $t$ , then the second condition of that proposition ensures that  $M$  does not contain entries in any row or column from both sides of the highlighted path. For a given  $m \in M$  which appears above the highlighted path in  $A$ , the first condition of the proposition ensures that all positive entries below and to the right of  $m$  also occur in  $M$ ; similarly for  $m \in M$  such that  $-m$  appears in  $A$ . Thus  $M$  corresponds to a path through  $A$  with all movements up or to the left.  $\square$

As the number of paths through the grid is  $\binom{\lfloor \frac{s}{2} \rfloor + \lfloor \frac{t}{2} \rfloor}{\lfloor \frac{s}{2} \rfloor}$ , we have completed the proof of the Theorem 1.

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