

LOGARITHMIC DIFFERENTIATION: TWO WRONGS MAKE A RIGHT

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You're teaching a Calculus class, and get to the point in the course where it is time to discuss logarithmic differentiation. As usual, you begin by asking the class "If $f(x) = x^x$, what is $f'(x)$?"

One student raises his hand and says "That's just the power rule. It's xx^{x-1} ." Another student says "That's just for when the exponent is constant. This has x in the exponent, so it's like e^x , but when it's not e you need to put the log in. So it's $(\ln x)x^x$." Smiling, you point out that just as one is assuming that n is a constant when one uses the formula for the derivative of x^n , one is assuming that a is a constant when one uses the formula for the derivative of a^x . Since neither the base nor the exponent of x^x is constant, the function $f(x) = x^x$ is neither a power function nor an exponential function, and therefore the derivative of x^x cannot be found using either of these rules. Instead, you say, we will use a technique called logarithmic differentiation.

(Pedagogical aside: Of course, here you have the option of using the definition $x^x = e^{x \ln x}$; you already have covered methods for differentiating this. But the technique of using logarithmic differentiation to break the natural log of a function into a sum of easily-differentiable summands is a nice one, and this is a very good context in which to introduce it.)

At this point a student in the back of the class raises her hand and says "Isn't $f(x) = x^x$ a combination of a power function and an exponential function, and therefore shouldn't the derivative be a combination of the derivative of a power function and the derivative of an exponential function?"

Trying not to discourage the student, you attempt to take her question seriously. You ask if she means that the derivative should be the sum of the two answers given by the first two students. She replies "sure."

You decide to indulge the student, saying "Let's see what happens. The sum of the two answers is $xx^{x-1} + (\ln x)x^x$. The real answer can be found as follows: First, we let $y = x^x$. Then we take the

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natural logarithm of both sides, obtaining $\ln y = \ln(x^x) = x \ln x$. Differentiating both sides of this equation with respect to x gives us

$$\frac{y'}{y} = (1)(\ln x) + x\left(\frac{1}{x}\right) = \ln x + 1,$$

and multiplying both sides by y yields

$$y' = y(\ln x + 1).$$

Substituting x^x for y , we see that the derivative y' of $y = x^x$ is

$$y' = x^x(\ln x + 1)."$$

You are about to say “As you can see, the answers are completely different,” when it occurs to you that they are not different at all! Flustered, you try to assure the class that it is just a coincidence, but you are not so sure.

After class, you hurry back to your office where you decide to see if this example of “two wrongs making a right” is indeed just a coincidence. What about functions of the form $f(x)^{g(x)}$? Reasoning as your student had above, you first treat the exponent $g(x)$ as a constant, obtaining $g(x)f(x)^{g(x)-1}f'(x)$, using the chain rule. Next, you treat $f(x)$ as a constant, obtaining $\ln(f(x))f(x)^{g(x)}g'(x)$, again using the chain rule.

Adding the two “wrongs” gives

$$(f(x)^{g(x)})' = g(x)f(x)^{g(x)-1}f'(x) + \ln(f(x))f(x)^{g(x)}g'(x),$$

which can be rewritten as

$$(1) \quad (f(x)^{g(x)})' = g(x)f(x)^{g(x)}\frac{f'(x)}{f(x)} + \ln(f(x))f(x)^{g(x)}g'(x).$$

You find the “true” derivative by writing $y = f(x)^{g(x)}$, taking natural logs, and differentiating both sides with respect to x :

$$y = f(x)^{g(x)}$$

$$\ln y = \ln(f(x)^{g(x)}) = g(x) \ln(f(x))$$

$$\frac{y'}{y} = g(x) \cdot \frac{1}{f(x)} \cdot f'(x) + g'(x) \ln(f(x))$$

$$y' = f(x)^{g(x)}\left(\frac{g(x)}{f(x)}f'(x) + g'(x) \ln(f(x))\right)$$

which simplifies to

$$(2) \quad y' = f(x)^{g(x)}g(x)\frac{1}{f(x)}f'(x) + f(x)^{g(x)}g'(x) \ln(f(x)).$$

You are simultaneously devastated and delighted to find that the expressions in (1) and (2) are equal!

It is clear now that it was not a coincidence that the “two wrongs made a right.” Instead, you realize that what the student wanted to do was indeed legitimate. But why does it work? Why can you first treat $g(x)$ as a constant, and then treat $f(x)$ as a constant? As soon as this thought enters your head, you realize the answer. If $y = f(x)^{g(x)}$, then y depends on two unknowns, f and g , which in turn depend on the parameter x . Thus to take the derivative of y with respect to x , you must calculate

$$\frac{\partial y}{\partial f} \frac{df}{dx} + \frac{\partial y}{\partial g} \frac{dg}{dx},$$

by the chain rule for a function of two variables (see [2, p. 673]. When calculating $\frac{\partial y}{\partial f}$, one treats $g(x)$ as if it were constant, and when one calculates $\frac{\partial y}{\partial g}$, one treats $f(x)$ as if it were constant. Since this is precisely what the student in your class was doing, you now realize that it is indeed a valid technique.

USEFULNESS?

The reader may believe that a situation such as the one described above is unlikely to occur. However, a situation in which a student actually did something equivalent on a test is related in [4], and is addressed in slightly greater generality in [1].

The standard sum, product, and quotient rules from first-semester calculus also lend themselves to a similar interpretation: To differentiate $f(x)g(x)$ (or $f(x) + g(x)$ or $f(x)/g(x)$), first differentiate as if $g(x)$ was constant, then as if $f(x)$ was constant; add the results. Indeed, a function $f(x)$ which is comprised of an algebraic combination of an arbitrary number of simpler functions $f_i(x)$ can be differentiated in this way, differentiating as if all the f_i are constant except for f_1 ; repeat for all f_j ; add the results.

This point of view on the standard “differentiation rules” might be a useful unifying idea. Students typically see differentiation rules as a disparate collection of formulas to memorize. At the least, pointing out this connection between the sum, product, quotient, and exponentiation rules raises the question of “why?” The answer — the proof of the multivariable chain rule — is a semester or two away, however. When it does arrive, these first-semester “rules” are nice examples to have ready.

A possible approach would be to teach the differentiation of functions of the form $y = f(x)^{g(x)}$ using this point of view. A possible

advantage would be that the students would not have to learn yet another technique (logarithmic differentiation), and could instead simply combine two formulas that they have already learned. However, the the technique of logarithmic differentiation is probably more important to most instructors than simply the ability to differentiate $f(x)^{g(x)}$.

This observation about differentiation rules does raise an interesting grading question: If a student writes “the derivative of x^x is xx^{x-1} ”, how much partial credit will you give? Shouldn’t you give the answer half credit, since it is half right?

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