Chapter Ten

Exponential Expressions and Functions

In Chapter 7 we studied one type of expression involving exponents: powers such as x^2 and x^3. In those expressions the exponent is a constant and the variable occurs in the base. In this chapter we consider exponential expressions such as 2^x and 3^x, in which the base is a constant and the variable occurs in the exponent.

Exponential expressions describe quantities that change by the same factor for each unit of time. For example, if a population doubles every year, if a bank account increases by 0.1% each month, or a mass of radioactive substance decreases by $1/2$ every 462 years, we expect an exponential expression.

In this chapter we see how to transform exponential expressions using the rules of exponents, and how to construct exponential functions that model phenomena such as population growth. The different forms of an exponential expression reveal different properties of the growth pattern, such as the annual percentage rate of change or the time it takes for the population to double.

Unlike other equations we have looked at, exponential equations cannot be solved using the basic operations we have introduced so far. In this chapter we see how to find qualitative information about solutions using the properties of exponents and how to estimate solutions to exponential equations graphically and numerically. In the next chapter we introduce a new operation for solving exponential equations.

10.1 EXPONENTIAL EXPRESSIONS

In an exponential expression, the variable occurs in the exponent. Exponential expressions arise when we have a quantity that changes by the same factor for each unit of time. For example, if a population doubles every year, if a bank account increases by 0.1% each month, or a mass of radioactive substance decreases by $1/2$ every 462 years, we expect an exponential expression.

Example 1 In a petri dish, there are $3 \cdot 2^t$ bacteria after t hours. How many bacteria are there initially? After one hour? After two hours? What is the meaning of the 2 and the 3 in this expression?

Solution Initially $t = 0$, so we have

$$\text{Initial population} = 3 \cdot 2^0 = 3.$$

One hour later, when $t = 1$, we have

$$\text{Population after 1 hour} = 3 \cdot 2^1 = 3 \cdot 2 = 6,$$

so the population doubles during the first hour. After two hours, when $t = 2$, we have

$$\text{Population after 2 hours} = 3 \cdot 2^2 = 3 \cdot 4 = 12.$$

Again, the population has doubled, which is the same as being multiplied by 2. The population is multiplied by 2 during any hour, because, starting at time t, we have

$$(\text{Population one hour later}) = 3 \cdot 2^{t+1} = 3 \cdot 2^t \cdot 2 = 2(3 \cdot 2^t) = 2\,(\text{Population at time } t).$$

Thus the 3 is the initial population and the 2 is the amount it gets multiplied by every hour.

In general

Exponential Expressions

An exponential expression in t has the form

$$a \cdot b^t, \quad a \text{ and } b \text{ constants, } b > 0$$

where

- a is the *initial* or *starting value*
- b is the *growth factor*, the amount by which the value of the expression gets multiplied for each unit increase in t.

We also call b the *base* of the exponential expression.

Exponential expressions can also describe quantities that are decreasing rather than growing.

Example 2 After the application of a bactericide, a population of bacteria is decreasing, and is given by

$$800 \cdot \left(\frac{3}{4}\right)^t$$

after t hours. Find the initial population and the population after one and two hours. What is the practical interpretation of the 800 and $3/4$ in this expression?

Solution We have

$$\text{Initial population} = 800 \cdot \left(\frac{3}{4}\right)^0 = 800$$
$$\text{Population after 1 hour} = 800 \cdot \left(\frac{3}{4}\right)^1 = 600$$
$$\text{Population after 2 hours} = 800 \cdot \left(\frac{3}{4}\right)^2 = 450.$$

The 800 is the starting population of bacteria, and the $3/4$ is the factor by which it gets multiplied every hour. Since $3/4 < 1$, the population decreases every hour.

When $b < 1$ in an exponential expression $a \cdot b^t$ then it represents a quantity that is decreasing or decaying, and we call b the *decay factor*.

Growth Factors and Percentage Growth Rates

Exponential growth is often expressed in terms of percentages rather than growth factors.

Example 3 Find the growth factor in one year for a quantity that

(a) Grows by 5% each year (b) Decays by 18% each year (c) Grows by 100% each year

Solution

(a) We start with an initial value a. First we calculate 5% of a, which is $0.05a$, then we add that to a, getting

$$\text{Amount after one year} = a + 0.05a = (1 + 0.05)a = 1.05a.$$

Thus the initial quantity is multiplied by 1.05 at the end of one year, so the growth factor is 1.05.

(b) This time we start with a, calculate 18% of a, which is $0.18a$, and subtract that amount from a, getting

$$\text{Amount after one year} = a - 0.18a = (1 - 0.18)a = 0.82a.$$

So the growth factor is 0.82, which in this case is a decay factor.

(c) Here we add 100% of a, which is just a, so

$$\text{Amount after one year} = a + a = 2a,$$

so the growth factor is 2.

In general, we have

$$\boxed{\text{Growth factor for } r\% \text{ growth rate} = 1 + \frac{r}{100}}$$

Example 4 The balance in a bank account in dollars, after t years, is given by

$$7000 \cdot 1.03^t.$$

Identify the initial value and the growth factor and explain what they mean in terms of the bank account.

Solution The initial value is the value when $t = 0$, so it is $7000 \cdot 1.03^0 = 7000$. This means that the account started with \$7000. The growth factor is 1.03. Since $1.03 - 1 = 0.03 = 3\%$, this corresponds to an annual interest rate of 3%.

Example 5 The quantity in grams of a radioactive sample after t days is given by

$$150 \cdot 0.94^t.$$

Identify the initial value and the growth factor and explain what they mean in terms of the sample.

Solution The initial value, when $t = 0$, is $150 \cdot 0.94^0 = 150$. This means we start with 150 g of material. The growth factor is 0.94, which gives growth rate of -6% per day, since $0.94 - 1 = -0.06$. We call a negative growth rate a *decay rate*, so we say that the substance is decaying by 6% per day.

Using Different Measures of Time

Sometimes, we know by how much a quantity changes over one period of time and we want to know the change over another period.

Example 6 Which is better, an account earning 1% per month or one earning 13% per year?

Solution The account that earns 1% a month has a monthly growth factor of 1.01. At the end of 12 months the account has grown by a factor of 1.01^{12}. Since $1.01^{12} = 1.127$, this account grows by 12.7% a year, so the account that earns 13% a year is slightly better.

Example 7 The population in thousands of a town after t years is given by

$$50(1.035)^t.$$

What is the town's annual growth rate? Monthly growth rate? Growth rate per decade?

Solution From the expression, we see that the growth factor is 1.035, so each year the population is 3.5% larger than the year before, so it is growing at 3.5% per year.

If m is the monthly growth factor then, at the end of the year, the population is m^{12} times what it was at the beginning of the year. But we know that it is 1.035 times what it was, so

$$\begin{aligned} m^{12} &= 1.035 \\ m &= 1.035^{1/12} = 1.0029. \end{aligned}$$

From this we see that the monthly growth rate is 0.29%.

At the end of a decade, 10 years, we know that the population is $50(1.035)^{10}$ or 1.41 times what it had been. This tells us that the growth rate per decade is 41%.

Exercises and Problems for Section 10.1

Exercises

In Exercises 1–4, write the expression either as a constant times a power of x, or a constant times an exponential in x. Identify the constant. If a power, identify the exponent, and if an exponential, identify the base.

1. ex^2 **2.** $3a^{2x}$ **3.** xb^2 **4.** 5^{x-1}

In Exercises 5–8 give the growth factor that corresponds to the given growth rate.

5. 8.5% growth **6.** 46% shrinkage

7. 215% growth **8.** 99.99% shrinkage

In Exercises 9–12 give the growth rate that corresponds to the given growth factor.

9. 1.7 **10.** 0.27 **11.** 5 **12.** 0.639

In words, give a possible interpretation of the expressions in Exercises 13–16.

13. $\$10{,}000(1.06)^t$ **14.** $(47 \text{ grams })(0.97)^x$

15. $(400 \text{ people})(1.006)^y$ **16.** $\$100{,}000(0.98)^a$

In Exercises 17–20 is the described rate of growth exponential? Explain.

17. The price of gas if it grows by \$0.02 a week.

18. The quantity of a prescribed drug in the bloodstream if it shrinks by 8.5% every 4 hours.

19. The speed of personal computers if it doubles every 3 years.

20. The height of a baseball thrown straight up into the air.

Suppose a, b, c are positive constants and t is a variable. Are the expressions in Exercises 21–28 equivalent? If so, explain why. If not, give a value for the variable t that leads to different values for the two expressions.

21. $a^2 \cdot a^t$ and a^{2+t} **22.** $b^t + 2b^t$ and $3b^t$

23. $a^3 \cdot a^{2t}$ and $a^6 t$ **24.** $6a^t$ and $(6a)^t$

25. 10^{-t} and $(0.1)^t$ **26.** $(2^t)^2$ and $2^{(t^2)}$

27. $2^t \cdot 3^{t+1}$ and 6^{2t+1} **28.** $12^t \cdot 3^{-t}$ and 4^t

Problems

29. The population of a city after t years is given by $220{,}000 \cdot 1.016^t$. Identify the initial value and the growth factor and explain what they mean in terms of the city.

30. After t years, an initial population P_0 has grown to $P_0(1+r)^t$, where r is the yearly growth rate. If the population doubles during the first year, which of the following are possible values of r?

(a) $r = 2\%$ **(b)** $r = 50\%$
(c) $r = 100\%$ **(d)** $r = 200\%$

The population of Austin, Texas, is increasing by 2% a year, that of Bismark, North Dakota is shrinking by 1% a year, and that of Phoenix, Arizona, is increasing by 3% a year. Interpret the quatities in Problems 31–33 in terms of one of these cities.

31. $(1.02)^3$ **32.** $(0.99)^1$ **33.** $(1.03)^2 - 1$

Prices are increasing at 5% per year. What is wrong with the statements in Problems 34–42? Correct the formula in the statement.

34. A \$6 item costs $\$(6 \cdot 1.05)^7$ in 7 years time.

35. A \$3 item costs $\$3(0.05)^{10}$ in ten years time.

36. The percent increase in prices over a 25-year period is $(1.05)^{25} \cdot 100$.

37. If time t is measured in months, then the price of a \$100 item at the end of one year is $\$100(1.05)^{12t}$.

38. If the rate at which prices increase is doubled, then the price of a \$20 object in 7 years time is $\$20(2.10)^7$.

39. If time t is measured in decades (10 years), then the price of a \$45 item in t decades is $\$45(1.05)^{0.1t}$.

40. Prices change by $10 \cdot 5\% = 50\%$ over a decade.

41. Prices change by $(5/12)\%$ in one month.

42. A \$250 million town budget is trimmed by 1% but then increases with inflation as prices go up. Ten years later, the budget is $\$250(1.04)^{10}$ million.

The expressions in Problems 43–50 involve several letters. Think of one letter at a time representing a variable and the rest as non-zero constants. In which cases is the expression linear? In which cases is it exponential? In this case, what is the base?

43. Ab^C

44. $P(1+r)^t$

45. $(P(1+r))^t$

46. $Ac^P - Bc^P$

47. $2^n a^n$

48. $a^n b^n + c^n$

49. $AB^q C^q$

50. Ab^{2t}

51. Let $Q = 8(0.87)^t$ give the level of a pollutant (in tons) remaining in a lake after t months. What is the monthly rate of decrease of the pollutant? The annual rate? The daily rate?

52. Let $P = 32(1.047)^t$ give the population of a town (in thousands) in year t. What is the town's annual growth rate? Monthly growth rate? Growth rate per decade?

53. Which is better, an account earning 2% per month or one earning 7% every 3 months?

54. A sunflower grows at a rate of 1% a day; another grows at a rate of 7% per week. Which is growing faster? Assuming they start at the same height, compare their heights at the end of 1 and 5 weeks.

10.2 EXPONENTIAL FUNCTIONS

If a quantity Q is given by an exponential expression in t,

$$Q = a \cdot b^t,$$

then we say that Q is an *exponential function* of t. As we saw in the last section, exponential functions describe quantities that are growing (or decaying) with a constant growth factor. In this section we see how to construct exponential functions.

Example 1 You put \$100 into an account earning 5% interest per year. How much do you have at the end of

(a) 1 year (b) 2 years (c) 3 years (d) 5 years (e) t years?

Solution

(a) After 1 year, the balance in dollars is given by

$$\begin{aligned} \text{Balance} &= \text{Original deposit} + \text{Interest} \\ &= 100 + 5\% \cdot 100 \\ &= 100 + 0.05 \cdot 100 \\ &= 100(1+0.05) \\ &= 100 \cdot 1.05 = 105 \text{ dollars.} \end{aligned}$$

The value 1.05 by which we multiply is the growth factor.

(b) Since after 1 year you have \$105, after two years the balance in dollars is given by

$$\begin{aligned} \text{Balance} &= \text{Balance at end of first year} + \text{Interest} \\ &= 105 + 0.05 \cdot 105 \\ &= 105(1+0.05) \\ &= 105 \cdot 1.05 = 110.25 \text{ dollars.} \end{aligned}$$

Notice that the growth factor is again 1.05. Thus

$$\text{Balance at end of 2 years} = (100 \cdot 1.05)1.05 = 100(1.05)^2.$$

(c) Since after 2 years you have $\$110.25$, the balance in dollars at the end of 3 years is given by

$$\begin{aligned}\text{Balance} &= \text{Balance at end of second year} + \text{Interest}\\ &= 110.25 + 0.05 \cdot 110.25\\ &= 110.25(1+0.05)\\ &= 110.25 \cdot 1.05 = 115.76 \text{ dollars.}\end{aligned}$$

Since the growth factor is again 1.05, we have

$$\text{Balance at the end of 3 years} = 110.25 \cdot 1.05 = 100(1.05)^2 1.05 = 100(1.05)^3.$$

(d) Extending the pattern, we see

$$\begin{aligned}\text{Balance at end of 4 years} &= 100(1.05)^4 = 121.55 \text{ dollars}\\ \text{Balance at end of 5 years} &= 100(1.05)^5 = 127.63 \text{ dollars.}\end{aligned}$$

(e) Similarly, we see

$$\text{Balance at end of } t \text{ years} = 100(1.05)^t \text{ dollars.}$$

How Do We Find the Growth Factor?

We can often find the growth factor of an exponential function by treating it as a variable and solving a power equation.

Example 2 An account growing at a constant percentage rate contains \$5000 in 2015 and \$8000 in 2025. What is its annual growth rate?

Solution Let b be the yearly growth factor. Since the account started with \$5000 in 2015 and grew to \$8000 in 2025, which is 10 years later, we have

$$8000 = 5000b^{10}.$$

Thus

$$\begin{aligned}\frac{8000}{5000} &= b^{10}\\ 1.6 &= b^{10}.\end{aligned}$$

Taking each side to the 1/10 power, we have

$$\begin{aligned}(b^{10})^{1/10} &= (1.6)^{1/10}\\ b &= 1.048.\end{aligned}$$

Thus the yearly growth rate is 4.8%.

Example 3 A lab has 200 grams of Germanium 15 days after it receives a sample. Another 22 days later (37 days after receiving the sample) the lab has 51.253 grams. What is the decay rate per day of Germanium? How much did the lab receive initially?

Solution The quantity, Q, remaining t days after the sample is received is given by

$$Q = ab^t.$$

When $t = 15$, we know $Q = 200$, so

$$200 = ab^{15}.$$

In addition, when $t = 37$, we know $Q = 51.253$, so

$$51.253 = ab^{37}.$$

Dividing the equations gives

$$\frac{51.253}{200} = \frac{ab^{37}}{ab^{15}}.$$

Canceling gives

$$0.2563 = b^{37-15}$$
$$b^{22} = 0.2563.$$

Taking the 1/22 power of both sides, we have

$$(b^{22})^{1/22} = (0.2563)^{1/22}$$
$$b = 0.93999.$$

Since $1 - 0.93999 = 0.06001$, the decay rate is about 6% per day.

To calculate a, the initial quantity, we solve

$$200 = a(0.93999)^{15}$$
$$\frac{200}{(0.93999)^{15}} = a$$
$$a = 506.0.$$

Thus the lab received 506 grams.

Doubling Time and Half-Life

The growth of populations is often expressed in terms of *doubling time*. The doubling time is the amount of time it takes for the population to double. For example, if you start with 100 bacteria in a culture with a doubling time of 3 hours, then after three hours you will have 200 bacteria, and 3 hours after that you will have 400 bacteria, and so on.

Example 4 A population has a doubling time of 25 years. What is the growth rate per year?

Solution Let a be the initial population, and let b be the growth factor per year, so

$$P = ab^t$$

is the population after t years. When $t = 25$ we have $P = 2a$, so

$$2a = ab^{25}$$
$$2 = b^{25}$$
$$b = 2^{1/25} = 1.028.$$

So the population is multiplied by 1.028 every year, and the growth rate is 2.8%.

If a quantity is decaying instead of growing, we can express the decay in terms of half-life. The half-life is the amount of time it takes for the quantity to decrease to half of its original amount. For example, if you start with 100 grams of a substance with a half-life of 62 days, after 62 days you will have 50 grams, and 62 days after that, you will have 25 grams, and so on.

Example 5 Hafnium has a half-life of 12.2 hours. What is the decay rate per hour?

Solution Let a be the initial quantity of hafnium and let h be the quantity remaining after t hours. We know

$$h = ab^t,$$

where b is the hourly decay factor. When $t = 12.2$, we have $h = a/2$, so

$$\begin{aligned} \frac{a}{2} &= ab^{12.2} \\ \frac{1}{2} &= b^{12.2} \\ b &= \left(\frac{1}{2}\right)^{1/12.2} = 0.9448. \end{aligned}$$

If r is the hourly decay rate,

$$\begin{aligned} 1 - r &= 0.9448 \\ r &= 1 - 0.9448 = 0.0552 = 5.52\%. \end{aligned}$$

So hafnium decays at a rate of 5.52% per hour.

Example 6 How much of 100 grams of a substance with a 62 day half-life remains after 1 day?

Solution Since there are 100 grams initially, the quantity, y, in grams left after t days is given by

$$y = 100b^t.$$

When $t = 62$, we know $y = 50$, so

$$\begin{aligned} 50 &= 100b^{62} \\ \frac{50}{100} &= b^{62} \\ b &= (0.5)^{1/62} = 0.9889. \end{aligned}$$

Thus, after 1 day, the quantity remaining is

$$y = 100 \cdot b^1 = 100(0.9889) = 98.89 \text{ grams.}$$

Sometimes we are given the percentage change over a fixed period of time. We can use this information to find the growth factor as well.

Example 7 A drug leaves the body at a rate of 32% every 12 hours. How much should you give a patient if there should be 100 mg in the body after 1 hour? 5 hours?

Solution If y is the quantity of drug, in mg, in the body after t hours, then $y = ab^t$. Since 32% is eliminated every 12 hours, 68% remains when $t = 12$. At that time, $y = 0.68a$, so

$$\begin{aligned} 0.68a &= ab^{12} \\ 0.68 &= b^{12} \\ b &= (0.68)^{1/12} = 0.9684 \end{aligned}$$

If we want y to be 100 after 1 hour, we put $t = 1$ and solve for a:

$$\begin{aligned} 100 &= a(0.9684)^1 \\ \frac{100}{0.9684} &= a \\ a &= 103.26 \text{ mg.} \end{aligned}$$

So we should give the patient about 103 mg. Similarly, if we want y to be 100 at $t = 5$ hours, we solve

$$\begin{aligned} 100 &= a(0.9684)^5 \\ \frac{100}{(0.9684)^5} &= a \\ a &= 117.42 \text{ mg.} \end{aligned}$$

So we should give the patient about 117 mg.

Exponential Equations

Often we want to know when an exponential function will attain a specific value. This leads to an exponential equation.

Example 8 A population of 5 bacteria doubles every day. When will there be 80 bacteria?

Solution We know that after one day, we have 10 bacteria; after 2 days, we have 20 bacteria; after 3 days, we have 40 bacteria; and after 4 days, we have 80 bacteria. Alternatively, the number of bacteria, P, on day t is given by

$$P = 5 \cdot 2^t$$

since we start with 5 and have two times as many each day. We want to find when there are 80, so we solve for t in the equation

$$80 = 5 \cdot 2^t.$$

Dividing by 5 we get

$$\begin{aligned} \frac{80}{5} &= 5 \cdot \frac{2^t}{5} \\ 16 &= 2^t. \end{aligned}$$

Since $16 = 2^4$, we see that $t = 4$ is a solution to the equation.

Example 9 A population is given by $P = 100 \cdot 2^{t/7}$, where t is in years. What is its doubling time?

Solution The initial population is 100, so the doubling time is the time it takes for the population to reach 200, which is the solution of the equation

$$100 \cdot 2^{t/7} = 200.$$

Dividing by 100 we get

$$2^{t/7} = 2.$$

Since $2 = 2^1$, we want the exponent on the left to equal 1, so $t = 7$ is a solution. Thus the doubling time is 7 years.

In Examples 8 and 9 we were lucky to get an equation whose solution we could guess. In Chapter we see how to use logarithms to solve general exponential equations exactly. For now there is some qualitative information about solutions of exponential equations that we can get by using the following facts:

- If $a > 1$, then $a^x > 1$ if x is positive and $a^x < 1$ if x is negative.
- If $0 < a < 1$, then $a^x < 1$ if x is positive and $a^x > 1$ if x is negative.
- For any $a > 0$, and any x, we have $a^x > 0$, and $a^0 = 1$.

Example 10 Say whether each equation could have a positive solution, a negative solution, a zero solution, or no solution: (a) $4^x = 2$ (b) $5 \cdot 4^x = 3$ (c) $4^t = 3 \cdot 5^t$ (d) $3 + 2^a = 1$

Solution

(a) Since 2 and 4 are both greater than 1, x must be positive.
(b) Dividing through by 5 we get $4^x = (3/5)$. Since $4 > 1$ and $3/5 < 1$, a solution would have to be negative.
(c) Dividing through by 5^t and using the exponent laws, we get $(4/5)^t = 3$. Since $4/5 < 1$ and $3 > 1$, a solution would have to be negative.
(d) Subtracting 3 from both sides we get $2^a = -2$, which is impossible, so there can be no solution.

Exercises and Problems for Section 10.2

Exercises

Write an expression representing the quantities in Problems 1–4.

1. A population at time t years if it is initially 2 million and growing at 3% per year.
2. The value of an investment which starts at \$5 million and grows at 30% per year for t years.
3. The quantity of pollutant remaining in a lake if it is removed at 2% a year for 5 years.
4. The cost of doing a project, initially priced at \$2 million, if the cost increases exponentially for 10 years.

In Exercises 5–12, find the formula for the exponential function going through the two points.

5. $(3, 4)$; $(6, 10)$
6. $(2, 6)$; $(7, 1)$
7. $(-6, 2)$; $(3, 6)$
8. $(-5, 8)$; $(-2, 1)$
9. $(0, 2)$; $(3, 7)$
10. $(1, 7)$; $(5, 9)$
11. $(3, 6.2)$; $(6, 5.1)$
12. $(2.5, 3.7)$; $(5.1, 9.3)$

Exercises 13–15 give a population's doubling time in years. Find its growth rate per year.

13. 50
14. 143
15. 14
16. The balance in a bank account earning interest at $r\%$ per year doubles every 10 years. What is r?
17. Lead-214 has a half-life of 27 minutes. What percent of the initial amount remains after 1 minute?
18. Bismuth-210 has a half-life of 5 days. What is the decay rate per day?

Problems

19. The average rainfall in Hong Kong in January and February is about 1 inch each month. From March to June, however, average rainfall in each month is double the average rainfall of the previous month.

(a) Make a table showing average rainfall for each month from January to June.
(b) Write a formula for the average rainfall in month n, where $2 \le n \le 6$ and January is month 1.
(c) What is the total average rainfall in the first six months of the year?

20. Between 1994 and 1999, the national health expenditures in the United States was rising at an average of 5.3% per year. The U.S. health expenditures in 1994 were 936.7 billion dollars.

(a) Express the national health expenditures, P, in billions of dollars, as a function of the year, t, with $t = 0$ corresponding to the year 1994.
(b) Use this model to estimate the national health expenditures in the year 1999. Compare this number to the actual 1999 expenditures, which were 1210.7 billion dollars.

21. The city of Baltimore has been declining in population for the last ten years. In the year 2000, the population of Baltimore was 651 thousand and declining at a rate of 1.2% per year. If this trend continues:

(a) Give a formula for the population of Baltimore, P, in thousands, as a function of years, t, since 2000.
(b) What is the predicted population in 2010?
(c) To two decimal places, estimate t when the population is 550 thousand.

22. The number of cell phone subscribers in the US has grown exponentially since 1990. There were 16 million subscribers in 1994 and 97 million subscribers in 2000. Find an expression for S, the number of subscribers (in millions) in the US t years after 1990. To the nearest million, how many subscribers are predicted for 2004 by this expression?

23. The number of asthma sufferers in the world was about 84 million in 1990 and 130 million in 2001. Let N represent the number of asthma sufferers (in millions) worldwide t years after 1990. Assuming N grows exponentially, what is the annual growth rate?

The US population is growing by about 1% a year. In 2000, it was 282 million. What is wrong with the statements in Problems 24–26? Correct the equation in the statement.

24. The population will be 300 million t years after 2000, where $282(0.01)^t = 300$.

25. The population, P, in 2020 is the solution to $P - (282 \cdot 1.01)^{20} = 0$.

26. The solution to $282(1.01)^t = 2$ is the number of years it takes for the population to double.

27. Match the statement (a)–(b) with the solutions to one or more of the equations (I)–(VI).

I. $10(1.2)^t = 5$	II. $10 = 5(1.2)^t$
III. $10 + 5(1.2)^t = 0$	IV. $5 + 10(1.2)^t = 0$
V. $10(0.8)^t = 5$	VI. $5(0.8)^t = 10$

(a) The time an exponentially growing quantity takes to grow from 5 to 10 grams.
(b) The time an exponentially decaying quantity takes to drop from 10 to 5 grams.

28. Match the statements (a)–(c) with one or more of the equations (I)–(VIII). The solution for t in the equation should be the quantity described in the statement. An equation may be used more than once. (Do not solve the equations.)

I. $4(1.1)^t = 2$	II. $(1.1)^{2t} = 4$
III. $2^t = 4$	IV. $2(1.1)^t = 4$
V. $4(0.9)^t = 2$	VI. $(0.9)^{2t} = 4$
VII. $2^{-t} = 4$	VIII. $2(0.9)^t = 4$

(a) The doubling time for a bank balance.
(b) The half-life of a radioactive compound.
(c) The time for a quantity growing exponentially to quadruple if its doubling time is 1.

29. A exponentially growing population quadruples in 22 years. How long does it take to double?

30. A radioactive compound decays to 25% of its original quantity in 90 minutes. What is its half-life?

31. The balance in a bank account grows by a factor of a each year and doubles every 7 years.

(a) By what factor does the balance change in 14 years? In 21 years?
(b) Find a.

In Problems 32–34, write a formula for the quantity described.

32. An exponentially growing population which doubles every 9 years.

33. The amount of a radioactive substance whose half-life is 5 days.

34. The balance in an interest-bearing bank account, if the balance triples in 20 years.

Without solving them, say whether the equations in Exercises 35–50 have a positive solution, a negative solution, a zero solution, or no solution. Give a reason for your answer.

35. $9^x = 250$

36. $2.5 = 5^t$

37. $7^x = 0.3$

38. $6^t = -1$

39. $7 + 2^y = 5$

40. $25 \cdot 3^z = 15$

41. $13 \cdot 5^{t+1} = 5^{2t}$

42. $(0.1)^x = 2$

43. $5(0.5)^y = 1$

44. $5 = -(0.7)^t$

45. $28 = 7(0.4)^z$

46. $7 = 28(0.4)^z$

47. $0.01(0.3)^t = 0.1$

48. $10^t = 7 \cdot 5^t$

49. $4^t \cdot 3^t = 5$

50. $(3.2)^{2y+1}(1 + 3.2) = (3.2)^y$

51. Assume $r > 0$. Without solving equations (I)–(IV) for x, decide which one has

(a) The largest solution
(b) The smallest solution
(c) No solution

I. $3(1 + r)^x = 7$ II. $3(1 + 2r)^x = 7$
III. $3(1 + 0.01r)^x = 7$ IV. $3(1 - r)^x = 7$

52. Assume that a, b, r are positive and that $a < b$. Consider the solution for x to the equation $a(1 + r)^x = b$. Without solving the equation, what is the effect of increasing each of a, r, b, while keeping each of the other two fixed? Does the solution increase or decrease?

(a) a **(b)** r **(c)** b

In Problems 53–60, decide for what values of the constant A the equation has

(a) A solution **(b)** The solution $t = 0$
(c) A positive solution

53. $5^t = A$

54. $3^{-t} = A$

55. $(0.2)^t = A$

56. $A - 2^{-t} = 0$

57. $6.3A - 3 \cdot 7^t = 0$

58. $2 \cdot 3^t + A = 0$

59. $A5^{-t} + 1 = 0$

60. $2(0.7)^t + 0.2A = 0$

61. Formulas I–III all describe the growth of the same population, with time, t, in years:

I. $P = 15(2)^{t/6}$ II. $P = 15(4)^{t/12}$ III. $P = 15(16)^{t/24}$

(a) Show that the three formulas are equivalent.
(b) What does formula I tell you about the doubling time of the population?
(c) What do formulas II and III tell you about the growth of the population? Give answers similar to the statement which is the answer to part (b).

62. Match each of the descriptions of the populations (a)–(e) with one of the formulas I–IV.

(a) Doubles every 20 years.
(b) Doubles every 10 years.
(c) Quadruples every 40 years.
(d) Quadruples every 20 years.
(e) Decreases by 75% every ten years.

I. $y = 53.1(2)^{t/10}$ II. $y = 0.07(2)^{-t/10}$
III. $y = 7.24(4)^{t/40}$ IV. $y = 7.32(4)^{-t/10}$

10.3 GRAPHICAL AND NUMERICAL BEHAVIOR OF EXPONENTIAL FUNCTIONS

The Graph of an Exponential

We begin by graphing $y = 3 \cdot 2^x$. We calculate y at three points and tabulate the results:

Table 10.1 *Values of* $y = 3 \cdot 2^x$

x	0	1	2
y	3	6	12

Notice that the y values double every time x increases by 1. Figure 10.1 shows the three points we have calculated, connected with line segments. Figure 10.2 shows more points corresponding to

$x = 0.5, 1.5$. Note that the graph looks smoother than Figure 10.1. If we use enough values of x (by using a graphing calculator, for example) we get a graph that appears to be smooth with no corners, as in Figure 10.3.

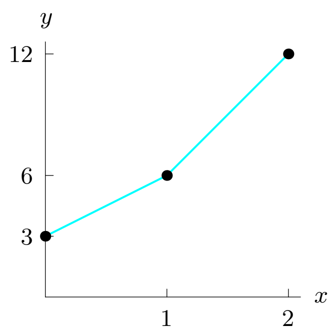


Figure 10.1: Connecting on integer values

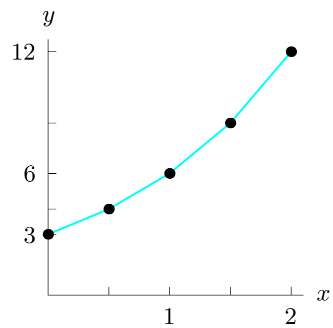


Figure 10.2: Connecting more points

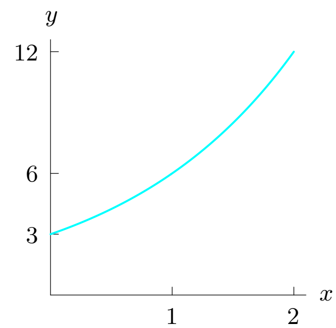


Figure 10.3: The graph of $y = 3 \cdot 2^x$

Example 1 Graph $y = 5(1.07)^x$ from $x = -10$ to $x = 10$.

Solution Notice that initial value, 5, is where the graph crosses the y-axis. Also, the y-values increase as we move from left to right, because the growth factor, 1.07, is greater than 1. Moreover, the graph increases more and more steeply as we move from left to right.

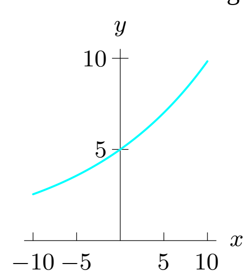


Figure 10.4: The graph of $y = 5(1.07)^x$

A growth factor that is less than 1 leads to a decreasing exponential.

Example 2 Graph $y = 22(0.8)^x$ from $x = -5$ to $x = 5$.

Solution Notice that the initial value, 22, is again the y-intercept. Also, this time the y-values decrease as we move from left to right, because the growth factor, 0.8, is less than 1.

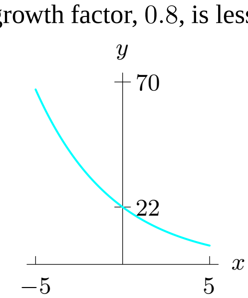


Figure 10.5: The graph of $y = 22(0.8)^x$

In general, we have

> For the graph of an exponential function $Q = ab^t$, the initial value a is the vertical intercept. The graph increases as we move from left to right if $b > 1$ and decreases if $0 < b < 1$.

Example 3 Find an equation to describe the exponential graph in Figure 10.6.

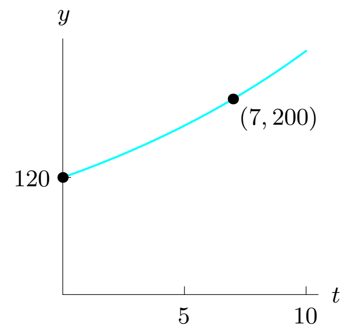


Figure 10.6

Solution We know that the formula is of the form $y = ab^t$, and we want to solve for a and b. The initial value, a, is the y-intercept (since $t = 0$ at the y-intercept), so $a = 120$. Thus

$$y = 120b^t.$$

Since we know that $y = 200$ when $t = 7$, we have

$$\begin{aligned} 200 &= 120b^7 \\ \frac{200}{120} &= b^7. \end{aligned}$$

Taking both sides to the $1/7$ power,

$$\begin{aligned} (b^7)^{1/7} &= \left(\frac{200}{120}\right)^{1/7} \\ b &= 1.076. \end{aligned}$$

Thus, $y = 120(1.076)^t$ is an equation for the curve.

Seeing Exponential Behavior in a Table

We saw in Table 10.1 that the y values doubled every time the x values increased by 1. In general, for a table of values that gives y as an exponential function of x and in which the x values increase by the same amount each time, the y values increase by a constant factor every time. We can check for this behavior by checking to see if the *ratio* of consecutive y-values is constant. If it is, the table could represent an exponential function.

Example 4 Could the values in Table 10.2 come from a formula $y = ab^x$?

Table 10.2

x	0	5	10	15
y	6	12	24	48

Solution Though in a table we don't know what happens between the values given, for every increase of x by 5, we see that y doubles

$$\frac{12}{6} = 2$$
$$\frac{24}{12} = 2$$
$$\frac{48}{24} = 2.$$

Therefore, this table could be the values of an exponential function. In fact $y = 6 \cdot 2^{x/5} = 6 \cdot (2^{1/5})^x$, which has the form $y = a \cdot b^x$ with $a = 6$ and $b = 2^{1/5}$.

Example 5 Table 10.3 shows values for an exponential equation. Find the equation.

Table 10.3

t	0	1	2	3
y	150	120	96	76.8

Solution Again, we know that $y = ab^t$. When $t = 0$, we have $y = 150$, so $a = 150$. Thus

$$y = 150b^t.$$

Since $y = 120$ when $t = 1$, we have

$$120 = 150b^1$$
$$b = \frac{120}{150} = 0.8.$$

Thus, the equation is $y = 150(0.8)^t$.

What if we don't know the y-intercept (the value of y when $t = 0$)? We can still find an equation from two points.

Example 6 Find the exponential equation for the data in Table 10.3 using only the last two data points.

Solution We still have $y = ab^t$, but we can no longer find a so easily. Using the last two points we have

$$96 = ab^2$$
$$76.8 = ab^3.$$

Dividing the two equations gives

$$\frac{76.9}{96} = \frac{ab^3}{ab^2}.$$

The right-hand side of the equation simplifies to b, so we have

$$0.8 = b.$$

Then we solve for a

$$96 = a(0.8)^2$$
$$\frac{96}{(0.8)^2} = a$$
$$150 = a.$$

Thus, the equation is $y = 150(0.8)^t$.

Using Tables and Graphs to Approximate Solutions to Equations

In Example 8 on page 206 we wanted to know when a population of bacteria reaches 80 if it is given by

$$P = 5 \cdot 2^t$$

after t days. Since

$$80 = 5 \cdot 16 = 5 \cdot 2^4$$

we could see directly that the solution is $t = 4$. What if we had wanted to find the time at which there were 70 bacteria? Since there are 40 bacteria after 3 days and 80 bacteria after 4 days, we know the answer must be between 3 and 4 days. How do we find the number more precisely?

One method is by trial and error—just trying different t values until you get as close as you want. Table 10.4 shows that $t \approx 3.81$ days. Another way is to look at a graph of $y = 5 \cdot 2^t$ and trace along it until the y-coordinate is 70. See Figure 10.7.

Table 10.4 *Bacteria after t days*

t	3	3.5	3.6	3.7	3.8	3.81	3.9	4
$y = 5 \cdot 2^t$	40	56.6	60.6	65.0	69.6	70.1	74.6	80

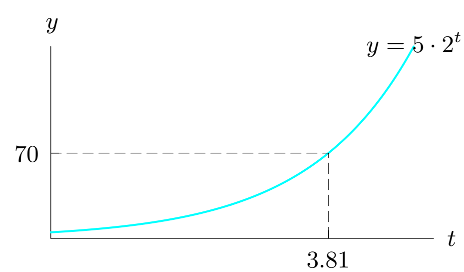


Figure 10.7: Graph of bacteria after t days

Example 7 A bank account contains \$22,000 and earns 3% interest per year. When will it contain \$62,000?

Solution After t years the balance is given by

$$B = 22{,}000(1.03)^t.$$

Figure 10.8 shows that when $B = 62{,}000$ we have $t \approx 35$. So it takes about 35 years for the balance to reach \$62,000.

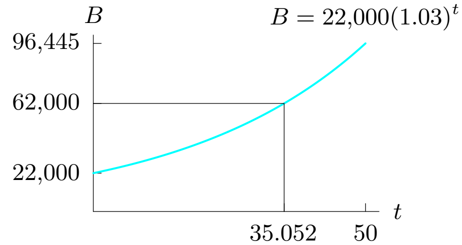


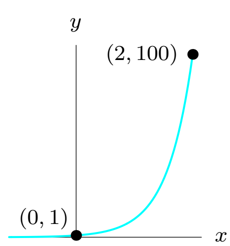
Figure 10.8: Graph of balance after t years

Exercises and Problems for Section 10.3

Exercises

For Exercises 1–6, find formulas for the exponential functions.

1.



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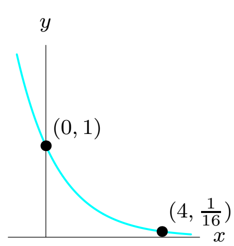

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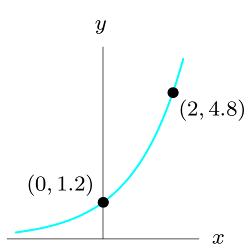

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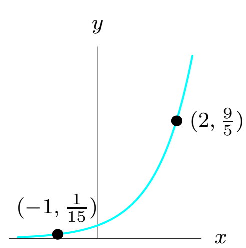

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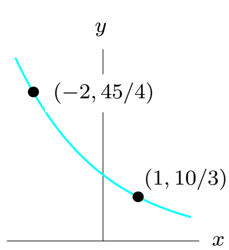

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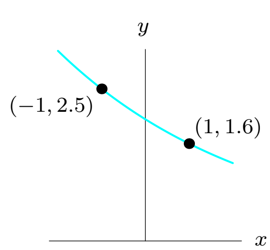


In Exercises 7–10, could the table give points on the graph of a function $y = ab^x$, for constants a and b? If so, find the function.

7.

x	0	1	2	3
y	1	7	49	343

8.

x	0	2	4	6
y	0	1	2	4

9.

x	3	6	9	12
y	4	4.4	4.84	5.324

10.

x	4	9	14	24
y	5	4.5	4.05	3.645

11. **(a)** Construct a table of $y = 28(1.1)^x$ for $x = 0, 1, 2, 3, 4$.
(b) For which values of x in the table is
(i) $y < 33.88$ (ii) $y > 30.8$
(iii) $y = 37.268$

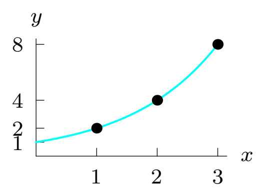
12. **(a)** Construct a table of $y = 526(0.87)^x$ for $x = -2, -1, 0, 1, 2$.
(b) Use your table to solve $604.598 = 526(0.87)^x$ for x.

13. **(a)** Construct a table of $y = 253(2.65)^x$ for $x = -3.5, -1.5, 0.5, 2.5$.
(b) At which x-values in your table is $253(2.65)^x \geq 58.648$?

Problems

Could the graphs in Exercises 14–19 be of exponential equations of the form $y = ab^x$?

14.



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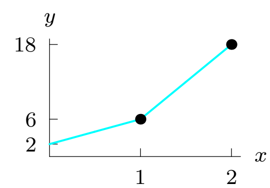

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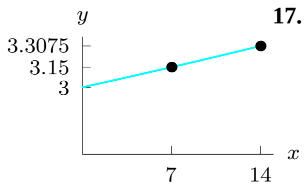
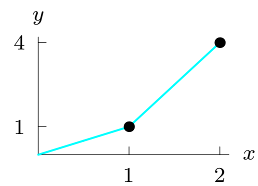
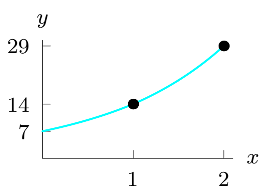


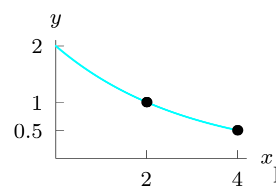
17.



18.



19.



20. Figure 10.9 shows the balance, P, in a bank account.

(a) Find a possible formula for P in terms of t assuming the balance grows exponentially.
(b) What was the initial balance?
(c) What annual interest rate does the account pay?

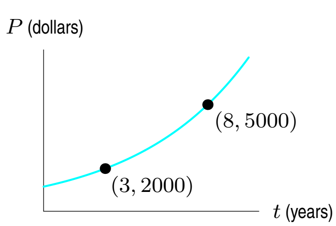


Figure 10.9

21. A 1987 treaty to protect the ozone layer produced dramatic declines in global production, P, of chlorofluorocarbons (CFCs). See Figure 10.10.[1] Find a formula for P as an exponential function of the number of years, t, since 1989. What was the annual percent decay rate?

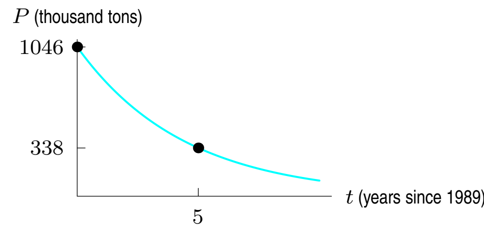


Figure 10.10

22. Graph $w = 8(1.16)^x$, $y = 20(1.01)^x$, and $z = 8(0.8)^x$ on the same axes from $x = -5$ to $x = 5$. Why does $y = 20(1.01)^x$ look like a straight line?

23. Sketch $y = 5(1.1)^x$, $y = 5(1)^x$, and $y = 5(0.9)^x$ on the same axis from $x = -3$ to $x = 3$. Discuss how changing the base changes the behavior of the graph.

In Problems 24–25 two exponential functions are given. Graph both on the same axes from $x = 0$ to $x = 5$. How does the growth factor affect the behavior of each graph?

24. $y = 5(1.05)^x$ and $y = 5(1.15)^x$

25. $y = 5(0.8)^x$ and $y = 5(0.6)^x$

In Problems 26–29, decide whether the values in the table could come from an exponential or linear formula, and then give a possible formula that fits the values.

26.

x	1	2	4	5
y	6	12	48	96

27.

x	0	3	6	9
y	5	10	15	20

28.

x	0	1	2	3
y	48	24	12	6

29.

x	0	2	4	6
y	3	9	27	81

[1] These numbers reflect the volume of the major CFCs multiplied by their respective ozone-depleting potentials (ODPs), as reported by the United Nations Environmental Programme Ozone Secretariat.

30. In year t, the population, L, of a colony of large ants is $L = 2000(1.05)^t$, and the population of a colony of small ants is $S = 1000(1.1)^t$.

(a) Construct a table showing each colony's population in years $t = 5, 10, 15, 20, 25, 30, 35, 40$.
(b) The small ants go to war against the large ants; they destroy the large ant colony when there are twice as many small ants as large ants. Use your table to determine in which year this happens.
(c) As long as the large ant population is greater than the small ant population, the large ants harvest fruit that falls on the ground between the two colonies. In which years in your table do the large ants harvest the fruit?

31. A lab receives a 1000 grams of an unknown radioactive substance that decays at a rate of 7% per day.

(a) Write an expression for Q, the quantity of substance remaining after t days.
(b) Make a table showing the quantity of the substance remaining at the end of $8, 9, 10, 11, 12$ days.
(c) For what values of t in the table is the quantity left
(i) Less than 500 gm?
(ii) More than 500 gm?
(d) A lab worker says that the half-life of the substance is between 11 and 12 days. Is this consistent with your table? If not, how would you correct the estimate?

32. From Figure 10.11, when is

(a) $100(1.05)^t > 121.55$
(b) $100(1.05)^t < 121.55$

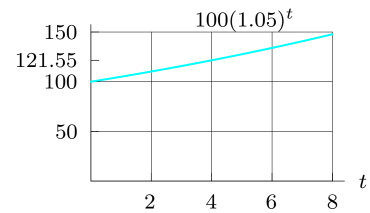


Figure 10.11

33. **(a)** Graph $y = 500(0.8)^x$ using the points $x = -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2$.
(b) Use your graph to solve
(i) $500(0.8)^x = 698.71$
(ii) $500(0.8)^x \geq 400$

34. **(a)** Graph $y = 250(1.1)^x$ and $y = 200(1.2)^x$ using the points $x = 1, 2, 3, 4, 5$.
(b) Using the points in your graph, for what x-values is
(i) $250(1.1)^x > 200(1.2)^x$
(ii) $250(1.1)^x < 200(1.2)^x$
(c) How might you make your answers to part (b) more precise?

REVIEW EXERCISES AND PROBLEMS FOR CHAPTER TEN

In Exercises 1–3, write the expression either as a constant times a power of x, or a constant times an exponential in x. Identify the constant. If a power, identify the exponent, and if an exponential, identify the base.

1. $\dfrac{-21}{2^x}$ **2.** $\dfrac{2^x}{3^x}$ **3.** $\dfrac{x^2}{x^3}$

In Exercises 4–7 give the growth factor that corresponds to the given growth rate.

4. 60% growth **5.** 18% shrinkage
6. 100% growth **7.** 99% shrinkage

As x gets larger, are the expressions in Exercises 8–12 increasing or decreasing? By what percent per unit increase of x?

8. $10(1.07)^x$ **9.** $5(0.96)^x$
10. $5(1.13)^{2x}$ **11.** $6\left(\dfrac{1.05}{2}\right)^x$
12. $\dfrac{5(0.97)^x}{3}$

In Exercises 13–16 give the rate of growth that corresponds to the given growth factor.

13. 1.095 **14.** 0.91 **15.** 2.16 **16.** 0.95

Identify the expressions in Exercises 17–22 as exponential, linear, or quadratic in t.

17. $7 + 2t^2$ **18.** $6m + 7t^2 + 8t$
19. $5q^t$ **20.** $7t + 8 - 4w$
21. $5h^n + t$ **22.** $a(7.08)^t$

In Exercises 23–27, say whether the quantity changing in an exponential or linear fashion.

23. An account receives a deposit of $723 per month.

24. A machine depreciates by 17% per year.

25. Every week, $9/10$ of a radioactive substance remains from the beginning of the week.

26. On liter of water is added to a trough every day.

27. After 124 minutes, $1/2$ of a drug remains in the body.

Problems

28. An investment is worth $500 \cdot 1.05^t$ dollars after t years. What was the initial investment? What was the annual interest rate?

29. In Examples 1 and 2 on page 198, describe the hourly change in population as a percentage.

30. A young sunflower grows in height by 5% every day; if the sunflower is 6 inches tall on the first day it was measured, write an expression that describes the height h of the sunflower in inches t days later.

31. A radioactive metal weighs 10 grams and loses 1% of its mass every hour. Write an expression that describes its weight after t hours.

32. A colony of bacteria is growing exponentially. At the end of 3 hours there are 1000 bacteria. At the end of 5 hours there are 4000.

(a) Write a formula for the population of bacteria at time t, in hours.
(b) By what percent does the number of bacteria increase each hour?

33. In 1940, there were about 10 brown tree snakes per square mile on the island of Guam, and in 2002, there were about 20,000 per square mile.[2] Find an exponential function for the number, N, of brown tree snakes per square mile on Guam t years after 1940. What was, on average, the annual percent increase in the population during this period?

In Exercises 34–37, could the function be linear or exponential or is it neither? Write possible formulas for the linear or exponential functions.

34.

r	1	3	7	15	31
$p(r)$	13	19	31	55	103

35.

x	6	9	12	18	24
$q(x)$	100	110	121	146.41	177.16

36.

x	10	12	15	16	18
$f(x)$	1	2	4	8	16

37.

t	1	2	3	4	5
$g(t)$	512	256	128	64	32

In Problems 38–40, correct the mistake in the formula.

38. A population which doubles every 5 years is given by $P = 7 \cdot 2^{5t}$.

39. The quantity of pollutant remaining after t minutes if 10% is removed each minute is given by $Q = Q_0(0.1)^t$.

40. The quantity, Q, which doubles every T years, has a yearly growth factor of $a = 2^T$.

41. For each description (a)–(c), select the formulas (I)–(VI) that could represent it. Assume P_0, r, and t are positive.

I. $P_0(1+r)^t$ II. $P_0(1-r)^t$
III. $P_0(1+r)^{-t}$ IV. $P_0(1-r)^{-t}$
V. $P_0t/(1+r)$ VI. $P_0t/(1-r)$

(a) A population increasing with time.
(b) A population decreasing with time.
(c) A population growing linearly with time.

42. A radioactive substance has a half-life of 25 years. What is its decay rate per year?

In Problems 43–45, say if 10^a is greater than or less than 10^b.

43. $0 < a < b$

44. $0 < a$ and $b < 0$

45. $a < b < 0$

In Problems 46–48 say if x^a is greater than or less than x^b.

46. $a < b$ and $0 < x < 1$

47. $a < b$ and $x > 1$

48. $x < 0$, a is an even integer, b is an odd integer

[2] *Science News*, Vol. 162, August 10, 2002, p. 85.

In Problems 49–51, say if $\frac{1}{10^a}$ is greater than or less than $\frac{1}{10^b}$.

49. $0 < a < b$

50. $0 < a$ and $b < 0$

51. $a < b < 0$

52. Without calculating them, put the following quantities in increasing order

(a) The solution to $2^t = 0.2$
(b) The solution to $3 \cdot 4^{-x} = 1$
(c) The solution to $49 = 7 \cdot 5^z$
(d) The number 0
(e) The number 1

53. A quantity grows or decays exponentially according to the formula $Q = ab^t$. Match the statements (a)–(b) with the solutions to one or more of the equations (I)–(VI).

I. $b^{2t} = 1$	II. $2b^t = 1$
III. $b^t = 2$	IV. $b^{2t} + 1 = 0$
V. $2b^t + 1 = 0$	VI. $b^{2t} + 2 = 0$

(a) The doubling time for an exponentially growing quantity.
(b) The half-life of an exponentially decaying quantity.

Without solving them, indicate the statement that best describes the solution x (if any) to the equations in Questions 54–63. *Example.* The equation $3 \cdot 4^x = 5$ has a positive solution because $4^x > 1$ for $x > 0$, and so the best-fitting statement is (a).

(a) Positive, because $b^x > 1$ for $b > 1$ and $x > 0$.
(b) Negative, because $b^x < 1$ for $b > 1$ and $x < 0$.
(c) Positive, because $b^x < 1$ for $0 < b < 1$ and $x > 0$.
(d) Negative, because $b^x > 1$ for $0 < b < 1$ and $x < 0$.
(e) There is no solution because $b^x > 0$ for $b > 0$.

54. $2 \cdot 5^x = 2070$

55. $3^x = 0.62$

56. $7^x = -3$

57. $6 + 4.6^x = 2$

58. $17 \cdot 1.8^x = 8$

59. $24 \cdot 0.31^x = 85$

60. $0.07 \cdot 0.02^x = 0.13$

61. $240 \cdot 0.55^x = 170$

62. $\frac{8}{2^x} = 9$

63. $\frac{12}{0.52^x} = 40$

64. The earth's atmospheric pressure, P, in terms of height above sea level is often modeled by an exponential decay function. The pressure at sea level is 1013 millibars and that the pressure decreases by 14% for every kilometer above sea level.

(a) What is the atmospheric pressure at 50 km?
(b) Estimate the altitude h at which the pressure equals 900 millibars.

65. A population of fish starts at 8000 in the year 2000 and decreases by 5.8% per year.

(a) Find a formula for the population, P, of fish t years after 2000.
(b) What population does the formula predict in the year 2005?
(c) Use a graph to estimate the year in which the population goes below 3000.

66. A container of ice cream is taken from the freezer and sits in a room for t minutes. Its temperature in degrees Fahrenheit is $a - b2^{-t} + b$, where a and b are positive constants. Write this expression in a form that

(a) Shows that the temperature is always less than $a + b$
(b) Shows that the temperature is always greater than a
(c) What are reasonable values for a and b?