

§2 Solving Equations I—Expressions Involving Power Functions

Toolbox—Combining Like Terms

Example 2.1 Simplify the expression $(x + 1)y + 4xy$ by combining like terms.

We can deconstruct the left part of the sum into the product of $x + 1$ and y . One way to deconstruct the right part of the sum is to split it into the product of $4x$ and y . Using this deconstruction, we see that y is a common term to each *product*, so $(x + 1)y$ and $4xy$ are *like terms*. To combine them, we factor out the common term and add or subtract whatever is left over. Here the leftover pieces of the like terms are expressions involving variables themselves—not just numbers. So what!

$$\begin{aligned} (x + 1)y + 4xy &= \underline{y} (x + 1) + \underline{y} 4x && \rightarrow \text{identify the common term,} \\ &= y[(x + 1) + 4x] && \rightarrow \text{factor out the common term,} \\ &= y(5x + 1) && \rightarrow \text{add the leftovers.} \end{aligned}$$

Example 2.2 Simplify the expression $x^2(4x + 3)^3 + x^3(4x + 3)^2$ by combining like terms.

This expression may not, at first glance, seem to have any like terms. But if you look back at Solution 2 of Example 1.3, both the left and right pieces of the sum have a common factor of $x^2(4x + 3)^2$. We use the same method as in the first two examples above:

$$\begin{aligned} &x^2(4x + 3)^3 + x^3(4x + 3)^2 \\ &= \underline{x^2(4x + 3)^2} (4x + 3) + \underline{x^2} x \underline{(4x + 3)^2} && \rightarrow \text{identify the common term,} \\ &= x^2(4x + 3)^2[(4x + 3) + x] && \rightarrow \text{factor out the common term,} \\ &= x^2(4x + 3)^2(5x + 3) && \rightarrow \text{add the leftovers.} \end{aligned}$$

Example 2.3 Simplify the expression $(x + 1)^{\cancel{2}}2x + x^2(\cancel{3}/2)(x + 1)^{\cancel{2}}$ by combining like terms.

We have to play with this expression a little bit to find out which deconstruction will be most useful to us. Right now you may be having trouble knowing the difference between what is useful and what is practically useless. What we optimally want is to identify all the common terms.

A useless way: Deconstruct the left-hand piece of the sum into the product of $(x + 1)^{\cancel{2}}$ and $2x$. Split the right hand piece of the sum into the product of $(\cancel{3}/2)x^2$ and $(x + 1)^{\cancel{2}}$. This deconstruction, though a perfectly legal (mathematically anyway) way to see the

expression, will not help us at all since we have not identified any common terms in the two products. Don't give up—try again!

A useful way: Look at the left-hand piece of the sum as the product of x and $2(x+1)^{\frac{1}{2}}$. Turn the right hand part of the sum into the product of x and $\frac{3}{2}(x+1)^{\frac{1}{2}}$. We have now found one common term—the x —so this is a good start.

An even more useful way: Notice that the leftover terms in the previous way each contain the component $x+1$, but raised to different powers. Use this! Since $(x+1)^1(x+1)^{\frac{1}{2}} = (x+1)^{\frac{3}{2}}$, we can turn the left-hand part of the sum into the product of $x(x+1)^{\frac{1}{2}}$ and $2(x+1)$. The right hand part of the sum can be written as the product of $x(x+1)^{\frac{1}{2}}$ and $(\frac{3}{2})x$. Thus we still see the common term of x , but also the common term of $(x+1)^{\frac{1}{2}}$.

We conclude the example by using this last deconstruction.

$$\begin{aligned}
 & (x+1)^{\frac{1}{2}}2x + x^2\frac{3}{2}(x+1)^{\frac{1}{2}} \\
 &= \underline{x(x+1)^{\frac{1}{2}}} 2(x+1) + \underline{x(x+1)^{\frac{1}{2}}} \frac{3}{2}x && \rightarrow \text{identify the common term,} \\
 &= x(x+1)^{\frac{1}{2}} 2(x+1) + \frac{3}{2}x && \rightarrow \text{factor out the common term,} \\
 &= x(x+1)^{\frac{1}{2}} 2x + 2 + \frac{3}{2}x && \rightarrow \text{add the leftovers.} \\
 &= x(x+1)^{\frac{1}{2}} \frac{7}{2}x + 2
 \end{aligned}$$

Note. In the preceding example, we stopped identifying common terms after we found the common term $(x+1)^{\frac{1}{2}}$. Could we have found any other common terms? How can you tell when you are done?

You can tell by looking at the leftover terms. In the example, $2(x+1)$ is leftover from the left-hand part of the sum, and $(\frac{3}{2})x$ is leftover from the right. Convince yourself that these leftover terms cannot be split into *products* that have a common term.

Common Mistake. A common error here is to think that x is a common term to each, and that we could factor out the x from both $2(x+1)$ and $(\frac{3}{2})x$. But we cannot write $2(x+1)$ as x something. The '+1' traps the x .

Toolbox—Zero Law

So what is the difference between an expression and an equation? The simple answer is that an equation has an '=' symbol. This symbol sets up a relationship between two expressions—our goal is to find out where this relationship is true. What this means is that our job is to find out for which values of the variable is the left-hand expression equal to the right-hand expression.

The Zero Law is a way to turn more complex equations into two or more simpler ones. Let ex_1 and ex_2 represent expressions (like $x + 1$ or $x^2 - 7x + 6$). If $ex_1 \cdot ex_2 = 0$, for a particular value of x , then either that value of x makes $ex_1 = 0$ or it makes $ex_2 = 0$ (or possibly both). This is very helpful to us because looking for the solutions of $ex_1 \cdot ex_2 = 0$ is the same as looking for the solutions of $ex_1 = 0$ and $ex_2 = 0$ separately. Of course we can do this with any number of expressions which are *multiplied* together.

Note that the Zero Law only works if one side of the equation is already zero! (Thus the name...)

Example 2.4 Use the Zero Law to find the solutions of $(x + 1)(x - 2)(x - 1) = 0$.

The expression on the left side is the product of three expressions. The Zero Law tells us that finding which x make the left-hand side equal to zero is the same as finding which x make at least one of the three pieces on the left equal to zero. We have:

$$\underbrace{(x + 1)}_{ex_1} \underbrace{(x - 2)}_{ex_2} \underbrace{(x - 1)}_{ex_3} = 0$$

The Zero Law tells us we can solve instead the three equations

$$\begin{aligned} x + 1 &= 0 & x &= -1 \\ x - 2 &= 0 & x &= 2 \\ x - 1 &= 0 & x &= 1. \end{aligned}$$

Applying the Toolbox—Solving Equations

Example 2.5 Solve the equation $-3x(x + 1) = -(x + 1)(x^2 + 2)$.

Guess and plug in method. The following table gives the outcome of plugging in various numbers into the left-hand and right-hand sides of this equation.

x	$-3x(x + 1)$	$-(x + 1)(x^2 + 2)$
-2	-6	6
-1	0	0
0	0	-2
5	-90	-162

☞ Here the value of the left-hand side equals the right-hand side.

So far, we've identified that when $x = -1$, the left-hand side of the equation equals the right-hand side. Are there any other values of x which do the same? How do you know? This is a very poor method.

Graphing method to estimate solutions. Instead of guessing which numbers to plug in, we could graph $y = -3x(x + 1)$ and $y = -(x + 1)(x^2 + 2)$ on the same set of axes to get a more complete picture. Finding the values of x which solve the equation is the same as finding the x -coordinates of the points where the graphs of $y = -3x(x + 1)$ and $y = -(x + 1)(x^2 + 2)$ intersect (or cross).

Use your graphing calculator to graph $y = -3x(x + 1)$ and $y = -(x + 1)(x^2 + 2)$ on the same axes. Change your viewing rectangle to make sure you are seeing all the important features of your graphs. Trace and zoom to estimate the x -coordinates of the points where the two graphs cross. Are any of your answers close to $x = -1$? Are there any other solutions?

Questions to ponder over the semester: How do you know that you have found all the solutions? Is your viewing rectangle large enough? These questions are not so easy to answer, but you will be able to by the end of the semester.

Using a graphing calculator will only allow you to *estimate* the solutions. We need to use algebra to find *exact* solutions. But using *both* graphing and algebra in the exercises will increase your understanding and allow you to check your answers.

Algebraic method. We'll use our toolbox in a series of steps.

Preliminary work. Get all terms which are on the right-hand side of the equation over to the left by adding or subtracting them to each side. With all your mental powers, *resist* the temptation to cancel out anything by dividing. You will only miss possible solutions. Here we add $(x + 1)(x^2 + 2)$ to each side to get

$$-3x(x + 1) + (x + 1)(x^2 + 2) = 0.$$

Combine all like terms. Deconstruct the left-hand side—play with it! How do you know when you have what you need? Experience and practice! Our end goal is to turn the left side into a *product* of expressions and then use the Zero Law.

One way to deconstruct the left part of the left-hand side is to see it as a product of $x + 1$ and $-3x$. The right part is the product of $x + 1$ and $x^2 + 2$. We have identified a common factor of $x + 1$:

$$\begin{aligned} -3x \underbrace{(x + 1)} + \underbrace{(x + 1)} (x^2 + 2) &= 0 && \rightarrow \text{identify the common term,} \\ (x + 1) \left[-3x + (x^2 + 2) \right] &= 0 && \rightarrow \text{factor it out.} \end{aligned}$$

Now at this stage we pause and check: are there any other common terms in the leftover pieces? There are no other common terms but if there were, we would factor them out in the same way. Continue the process of identifying common terms and factoring them out, until there are no more. Then add or subtract the leftovers. Our equation now looks like

$$(x + 1)(x^2 - 3x + 2) = 0.$$

This quadratic term can be factored (always check for this), so we end up with

$$(x + 1)(x - 2)(x - 1) = 0.$$

Use the Zero Law. We did this part of the problem in Example 2.4. The Zero Law says that the solutions of $(x + 1)(x - 2)(x - 1) = 0$ are the same as the solutions of $x + 1 = 0$, $x - 2 = 0$, and $x - 1 = 0$. Thus our final answers are $x = -1, 1, 2$.

How do these solutions compare to the answers you estimated with your graphing calculator?

Example 2.6 Solve the equation

$$\frac{(x - 4)^2 \frac{5}{2}(x + 2)^{\cancel{2}} - (x + 2)^{\cancel{2}} 2(x - 4)}{(x - 4)^4} = 0.$$

Graphing Method. Following the previous example we see that we want to graph the left-hand side and the right-hand side on the same set of axes. We can estimate the solutions to this equation by tracing and zooming in on the points of intersection. Try this in this case. Remember that $y = 0$ is just the x -axis. What happens near $x = 4$? In fact, your calculator is confused—it has no idea what to do at $x = 4$.

Using our brains, we see that plugging in $x = 4$ gives a zero denominator. This means that the left-hand side of the equation is *undefined* at $x = 4$.

We can graph the left-hand side with our graphing calculators as long as we have the calculator avoid $x = 4$. To do this, we must graph the function in two pieces—first with x values smaller than 4, and then with x values larger than 4.

Step 1: Graph $y = 0$ and

$$y = \frac{(x - 4)^2 \frac{5}{2}(x + 2)^{\cancel{2}} - (x + 2)^{\cancel{2}} 2(x - 4)}{(x - 4)^4}$$

on the same set of axes using a viewing window with a *maximum* x -coordinate of 3.5 or 3.99 (or anything just a bit smaller than 4).

Step 2: Graph these same functions using a window with a *minimum* x -coordinate of 4.01 or 4.1 (or something just a bit larger than 4).

In each step trace and zoom to estimate the x -coordinates of any place where

$$y = \frac{(x-4)^2 \frac{5}{2} (x+2)^{\frac{3}{2}} - (x+2)^{\frac{3}{2}} 2(x-4)}{(x-4)^4}$$

crosses the x -axis (which happens to be the second function $y = 0$).

Algebraic Method. Here we already have everything but zero on the left-hand side. Since the left-hand side contains algebraic fractions, we want to multiply through (both sides of the equation) by the common denominator to get rid of the algebraic fractions. In this case the common denominator is $(x-4)^4$. We now want to solve this *revised* equation.

Any x -value which is a solution of the original equation will be a solution to this revised equation. This means that we will find the solutions to the original equation among those to the revised equation

$$(x-4)^2 \frac{5}{2} (x+2)^{\frac{3}{2}} - (x+2)^{\frac{3}{2}} 2(x-4) = 0.$$

Read that last sentence carefully—a very important fact is that in solving the revised equation, we may get x -values which are *not* solutions to the original equation. So we have a two-step process: solve the revised equation, then check to see which of these values are *really* solutions to the original equation. This last step is called *checking for extraneous solutions*.

Solve revised equation. Solve the equation $(x-4)^2 \frac{5}{2} (x+2)^{\frac{3}{2}} - (x+2)^{\frac{3}{2}} 2(x-4) = 0$.

This is one complicated algebraic expression subtracted from another. When your equation is of this form, combining like terms will often help simplify it. We will follow the form of the examples in the Toolbox.

First notice that there is a factor of both $x-4$ and $x+2$ in each side of the minus symbol. We will take the smallest power of each of these terms as our common term. The smallest power of $x-4$ is 1 and the smallest power of $x+2$ is $\frac{3}{2}$. With the common terms underlined, our equation looks like

$$\underline{(x-4)} \underline{(x-4)} \frac{5}{2} \underline{(x+2)^{\frac{3}{2}}} - \underline{(x+2)^{\frac{3}{2}}} \underline{(x+2)} 2 \underline{(x-4)} = 0.$$

We now factor out the common terms and subtract the leftovers in the following series of steps.

$$(x+2)^{\frac{2}{3}}(x-4)(x-4)^{\frac{5}{2}} - (x+2)2 = 0$$

$$(x+2)^{\frac{2}{3}}(x-4)^{\frac{5}{2}}x - 10 - 2x - 4 = 0$$

$$(x+2)^{\frac{2}{3}}(x-4)^{\frac{5}{2}}\frac{1}{2}x - 14 = 0.$$

At this point the left-hand side of the equation is now the product of three simpler equations. Use the Zero Law! The solutions to this equation are the solutions to the three equations

$$x - 4 = 0$$

$$(x+2)^{\frac{2}{3}} = 0$$

$$\frac{1}{2}x - 14 = 0.$$

The first and third are linear equations yielding the solutions $x = 4$ and $x = 28$ respectively. The middle equation can be solved by raising each side to the power $\frac{3}{2}$, which is the reciprocal of the power $\frac{2}{3}$. Thus its solution is $x = -2$

Remember our two-step process! These x -values may be the solutions to the revised equation, but they may not even be in the domain of the left-hand side function. We have to worry about zeros in the denominator as well as negative numbers under a square root, fourth root, etc.

Checking for extraneous solutions. The easiest way to do this is to just start plugging in to the *original equation* the (possibly false) solutions that you have already obtained. When you plug in an x -value, do you really get zero on the left? Or must you conclude the left-hand side is undefined at that x -value? We see that if we plug in $x = -2$ and $x = 28$ to the original equation we really do get $0 = 0$. But we have already seen that $x = 4$ gives a zero denominator. So our solutions to the original equation are only $x = -2, 28$. How do these check out with the solutions you estimated with the graphing method?

Summary: Solving Equations

Preliminary work. Get all terms to the left-hand side of the equation, leaving only zero on the right-hand side. Convert all negative exponents into positive exponents. If you have algebraic fractions, multiply through by the common denominator, and continue to the next step to solve this revised equation.

Tips on solving equations without algebraic fractions. It generally helps to keep asking yourself the following series of questions in order over and over again. If you answer yes to any of them, follow the instructions.

- Q1. Basic Questions: Is the left-hand side of the equation a linear, quadratic, or other polynomial that can be factored? If yes, ...you know what to do. Is the *entire* left-hand side raised to a power? If yes, raise each side to the reciprocal of that power to get rid of it, and then repeat question Q1.
- Q2. Is the left-hand side of the equation a product of algebraic expressions? If yes, use the Zero Law to break the original equation into smaller equations. Repeat the question process (starting with Q1) to solve *each* smaller equation.
- Q3. Are there any like terms? If yes, combine them following the examples in the toolbox. Go back to Q2.

If these steps do not lead to a complete solution, you should try anything else you can think of. But you might have to rely on the numerical estimates you obtained from graphing (or some other source).

Check for extraneous solutions. Plug in to the original equation all solutions obtained above. Do you get zero in the denominator? Or a negative number under a square root, fourth root, etc? Later on in the course, we will see other ways extraneous solutions will occur. Any solution which gives any kind of problem like this is NOT really a solution. All other solutions are your answers. Check them against the graphing estimates.