Confusing Clocks

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This paper had its genesis in the early 1980's, when the third author was teaching at a community college in California and was asked a version of the following question by a student:

> Given a standard analog (two-hand) clock, are there times when the two hands could be interchanged to obtain another valid time (besides the obvious times when the two hands are at the same position?

It was not hard to work out an answer to the question (see below), but the problem suggests many similar (and harder) questions. The question sat for years until the first author suggested it to one of his undergraduate students, the second author.

The most obvious of these questions regards a three-hand clock: Given a standard three-hand clock (with hour, minute, and second hands), are there times when the hands could be permuted in some way to obtain another valid time? Again, overlapping hands provide trivial solutions. We examine this question in the second section; in the last section we consider imperfect clocks.

The two-hand problem also appears in [1, 2, 3, 4], with [3] giving the solution we give, [2, 4] giving algebraic solutions, and [1] giving hints towards the solution below.

The Two-Hand Clock

To start off, let us examine the case of a two-hand, perfectly accurate, twelve hour clock. As an example, take the time 2:00, when the hour hand points at 2 o'clock and the minute hand points at the 12 o'clock position. Permuting the hands, we do not get a valid clock position, since the hour hand pointing directly at 12 forces the minute hand to also point to 12. We notice that the position of the hour hand determines the position of the minute hand

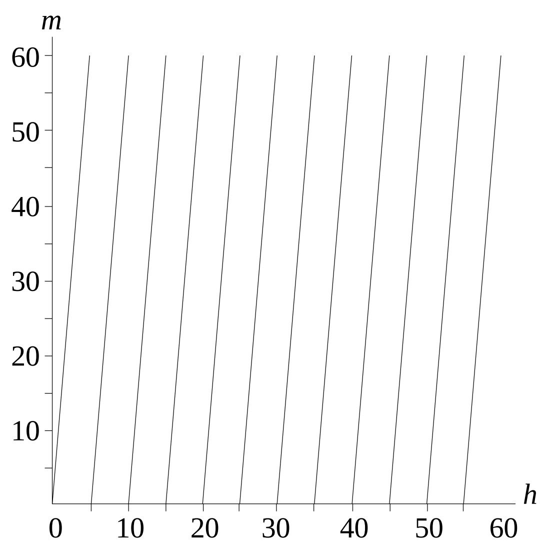

Figure 1: The graph of $m(h)$

— so we can write the minute hand position as a function of the hour hand position. If we use the 60-minute scale on the clock face (so we measure the position of each hand as a real number in the interval $[0,60)$ — this is the usual scale for the minute hand, but not for the hour hand), and use h to represent the hour hand and $m(h)$ the minute hand, we have:

$$\begin{aligned} m(h) &\equiv 12h \pmod{60} \\ &= 12h - 60[h/5], \end{aligned}$$

where $[x]$ means the greatest integer less than or equal to x. We will often indicate hand positions as ordered pairs (x,y), where x is the position of the hour hand and y the minute hand.

To answer the original question, we must find values of h for which $(m(h),h)$ is a valid clock position — i.e., for which $h = m(m(h))$. The last equation can be solved algebraically; this was done in [4], which includes an exhaustive list of all solutions. However, there is a nice way to "see" the answer (which appears in [3]): The point (a,b) in the plane represents one of the hand positions we are looking for if and only if (a,b) and (b,a) are both on the graph of the function $m(h)$, which has $[0,60)$ for its domain and whose graph is shown in Figure .

The point (b,a) is on this graph if and only if (a,b) is on the graph $(m(h),h)$, which is the reflection about the line $m = h$ of the graph above. Overlaying the two graphs, we have the graph shown in Figure , and the intersections are precisely the points we are looking for. There are 143 of them (the apparent intersection at $(60,60)$ is the same as the one at $(0,0)$); as mentioned above, they are catalogued in [4]. The 11 intersections which lie on the line $m = h$ are the trivial solutions where the hands are at the same position, so there are 132 non-trivial solutions.

Before discussing the three-hand clock, we mention one property of the greatest integer

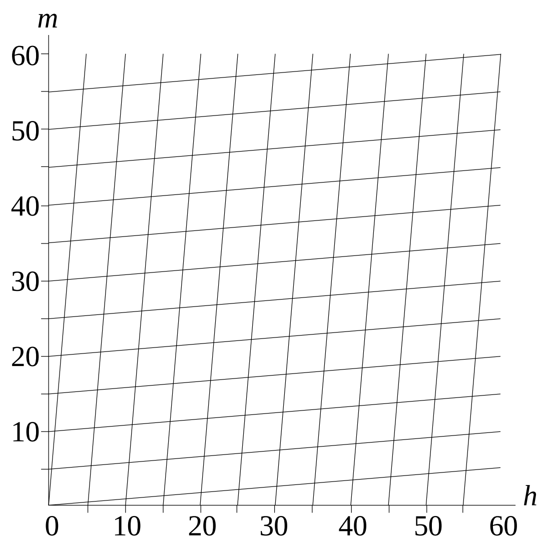

Figure 2: $m(h)$ and its reflection about the line $m = h$

function which we use frequently: If r is a real number and a an integer, then $[r+a] = [r]+a$

Three-Hand Clocks

Next consider a perfectly accurate three-hand clock. Our method of graphical intersections may not be so useful in the three-hand case. In this case the lines representing the time are of course in three dimensions and it is not necessarily true that non-parallel lines will intersect.

As with the minute hand, the position h of the hour hand determines the position of the second-hand on our clock, via the function

$$\begin{aligned} s(h) &\equiv 720h \pmod{60} \\ &= 720h - 60[12h]. \end{aligned}$$

With the additional hand on the clock come additional possible permutations of the hands. We represent hand positions as ordered triples (x,y,z), with z the position of the second-hand. There are six possible permutations of the hands:

1. $(h, m(h), s(h))$;
2. $(m(h), h, s(h))$;
3. $(h, s(h), m(h))$;
4. $(s(h), m(h), h)$;
5. $(s(h), h, m(h))$; and
6. $(m(h), s(h), h)$.

The first is the normal position of the hands and always represents a valid time. The only obvious value for h which gives a valid time for any of the other permutations is the trivial one $h = 0$, or 12:00:00. Are there others?

Two-Hand Switches. The second permutation is just the hour hand–minute hand switch discussed above ($h = m(m(h))$), with the additional requirement that the two hands have the second-hand in the same position ($s(h) = s(m(h))$). We examine the tables in [4] and see that there are no such solutions other than those 11 times when $h = m(h)$ — that is, the hour and minute hands overlap.

The third permutation leaves the hour hand alone, and since its position determines the other two, the only solutions here are when $s(h) = m(h)$; i.e., once again, when the hands overlap. These are not particularly interesting cases. It is not hard to find them algebraically; we simply note that the second-hand crosses the minute hand almost once a minute — 59 times every hour, to be precise — so there are $12 \cdot 59 = 708$ such overlaps per rotation of the hour hand.

The fourth permutation (and the last two-hand switch to consider) is harder, though one might notice that mathematically this is the same as the second permutation, with the roles of $s(h)$ and $m(h)$ reversed. We must simultaneously solve $h = s(s(h))$ and $m(h) = m(s(h))$. Here, however, there are $720^2 - 1 = 518399$ solutions to the first equation; compiling a table and looking in it for solutions to the second didn't sound like much fun. Note that the second-hand crosses the hour hand $12 \cdot 60 - 1 = 719$ times in one revolution of the hour hand, so there are at least that many solutions.

We solve $m(h) = m(s(h))$ first. For convenience, let $\alpha = h/5$ (α ranges from 0 to 12). Straightforward calculation shows that

$$\begin{aligned} m(h) = m(s(h)) \quad &\Leftrightarrow \quad 12h - 60[h/5] = 12(720h - 60[12h]) - 60[(720h - 60[12h])/5] \\ &\Leftrightarrow \quad [144h - [h/5]] = 719h/5 \\ &\Leftrightarrow \quad [720\alpha - [\alpha]] = 719\alpha. \end{aligned}$$

Clearly 719α must be an integer for this last equation to be true; conversely, if 719α is an integer, then $[720\alpha - [\alpha]] = [719\alpha + \alpha] - [\alpha] = 719\alpha + [\alpha] - [\alpha] = 719\alpha$. So the last equation is true if and only if 719α is an integer.

So we need α in $[0, 12)$ such that 719α is an integer. As 719 is prime, α must be of the form $n/719$ for some integer n satisfying $0 \le n < 12 \cdot 719 = 8628$. Now we check which of these 8628 solutions to $m(h) = m(s(h))$ also satisfies $h = s(s(h))$. Let $\alpha = \frac{n}{719}$ ($0 \le n \le 8627$) be such a solution. With some manipulation, the requirement $h = s(s(h))$ reduces to $60[12 \cdot 720h] = (720^2 - 1)h$. Now substitute 5α for h in this equation:

$$\begin{aligned} 60[12 \cdot 720 \frac{5n}{719}] &= (720^2 - 1)\frac{5n}{719} \\ [12 \cdot 720 \frac{5n}{719}] &= \frac{(720-1)(720+1) \cdot 5n}{60 \cdot 719} \\ &= \frac{721n}{12}. \end{aligned}$$

The right side of this last equation is an integer only if $12|n$. There are 719 multiples of 12 in the interval $[0, 8628)$; as we already know there are at least this many solutions, this is all of them. Once again, there are no non-trivial solutions.

This hand-switch can also be approached using *Mathematica* to do some of the arithmetic, but *Mathematica* doesn't save as much work on this permutation as it does on the cyclic permutation below.

Three-Hand Cyclic Permutations The two permutations we haven't considered yet are those which move all three hands. It will turn out that we only need to solve one of these; the other will be a consequence. Consider the fifth permutation above. If $(h, m(h), s(h))$ represents a valid time, then the requirements for $(s(h), h, m(h))$ to also represent a valid time are $h = m(s(h))$ and $m(h) = s(s(h))$.

We were getting tired of doing mod 60 algebra by hand at this point, so we asked *Mathematica* for help. *Mathematica* was not a big fan of non-prime moduli for modular arithmetic either, but we finally settled on the "Reduce" command, handling the modulus ourselves. This command simplifies equations, attempting to solve for the variable(s) we specify (`h` in the example below); the equations generated by "Reduce" are equivalent to the original equations and contain all possible solutions. If we define $m(h) = 12h$ and $s(h) = 720h$, then the command

```
In[3]:= Reduce[m[s[h]] - 60 k == h && m[h] == s[s[h]] - 60 j, h]
Out[3]= 8639 j = 518388 k && h = (60 k)/8639
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did the trick: Keeping in mind that j and k must be integers, and checking (again with Mathematica) that 518388 and 8639 are relatively prime, we find that $518388|j$ and $8639|k$. Since $8639|k$, the equation $h = \frac{60k}{8639}$ reduces modulo 60 to $h \equiv 0$. In other words, the only solution is the trivial one when all three hands overlap at 12:00:00.

Now, why don't we have to consider the other permutation? Let's say we had a solution for the sixth permutation $(m(h), s(h), h)$; that is, some value h_0 such that $s(h_0) = m(m(h_0))$ and $h_0 = s(m(h_0))$. We claim that then h_0 is a solution for the permutation we considered above! We must show that $h_0 = m(s(h_0))$ and $m(h_0) = s(s(h_0))$:

$$h_0 = s(m(h_0)) \equiv 720 \cdot 12h_0 \equiv m(s(h_0)) \pmod{60}.$$

Since h_0 and $m(s(h_0))$ are both non-negative and strictly less than 60, we conclude that $h_0 = m(s(h_0))$. Note that in fact we just showed that $s(m(h)) = m(s(h))$ for any h! Now

$$s(s(h_0)) = s(m(m(h_0))) = m(m(s(h_0))) = m(h_0).$$

This takes care of the last permutation.

We have shown that for a perfect three-hand clock, there are no times when hands can be interchanged to obtain valid clock positions, except for the obvious ones when the hands overlap. In other words, if you've a sharp eye, you can always tell what time it is on such a clock, even if the hands are installed in some permuted order.

Is your clock perfect?

Ours aren't. How strict must manufacturing tolerances be for the result to hold true? Let's investigate.

We'll assume that the spindles which turn the hands are geared together accurately, so we're not worried about the relative speeds of the hands. Our concern is with the proper alignment of the hands. Notice that there are 11 times in a 12-hour period at which the minute hand and hour hand will overlap. If we take one of these and turn the clock face so that the minute and hour hands are pointing at 0 (12), we see that any strange mounting of the hands can be considered as a mis-mounting of the second-hand only, with the minute and hour hands mounted perfectly.

With these assumptions, we have the following function for the second-hand in terms of the hour hand:

$$\begin{aligned}\hat{s}(h) &= 720h + o - 60[12h + o/60] \\ &\equiv 720h + o \pmod{60},\end{aligned}$$

where o stands for "offset". The function which describes the position of the minute hand is still $m(h) = 12h - 60[h/5]$. We want to find the minimal value of o which gives a non-trivial solution for one of the five non-trivial permutations in the last section.

It is immediate from the work done for the perfect clock that permutations 2 and 3 give no non-trivial solutions for any values of o. Permutation 4 is again more work, but the same argument as for the perfect clock again does the trick, with o added to the appropriate places in the calculations (that is, with $\hat{s}$ replacing s).

Finally, we have to consider the permutations that move all three hands. We again claim that we need only consider one of them. Assume h_0 is a value such that $(m(h_0), \hat{s}(h_0), h_0)$ is a valid time — that is,

$$\begin{aligned}h_0 &= \hat{s}(m(h_0)) \\ \hat{s}(h_0) &= m(m(h_0)).\end{aligned}$$

It is not true now that $(\hat{s}(h_0), h_0, m(h_0))$ is a valid time, but we claim that $(\hat{s}(12h_0), 12h_0, m(12h_0))$ is — so $12h_0$ gives the desired solution for the other permutation (with $12h_0$ reduced modulo 60 if necessary). We must show that $12h_0 = m(\hat{s}(12h_0))$ and $m(12h_0) = \hat{s}(\hat{s}(12h_0))$.

$$\begin{aligned}12h_0 = 12\hat{s}(m(h_0)) &\equiv m(m(m(m(h_0)))) \equiv m(\hat{s}(12h_0)) \pmod{60}; \\ m(12h_0) = m(m(h_0)) &= \hat{s}(h_0) = \hat{s}(\hat{s}(m(h_0))) = \hat{s}(\hat{s}(12h_0)).\end{aligned}$$

So we consider only the permutation $(m(h), \hat{s}(h), h)$. We want to find the smallest $|o|$ which will give a non-trivial hand-switch. We proceed with the help of Mathematica as for the perfect clock (s and m are the same functions as before: $s(h) = 720h$, $m(h) = 12h$).

```
In[3]:= Reduce[s[m[h]]+o-60 m == h && m[m[h]]==s[h]+o-60 j,{h,o}]
             -60 (j - m)          60 (8639 j - 576 m)
Out[3]= h == ----------- && o == -------------------
                8063                     8063
```

Working with this output, we find that 8639 and 576 are relatively prime; therefore there are integers m and j such that $8639j - 576m = 1$. As this will give us the smallest value for $|o|$, we find them (using the Euclidean algorithm or Mathematica) and get $m = 8624$, $j = 575$. Substituting these into $h = (60(-j+m))/8063$, we obtain $h = (60 \cdot 8049)/8063$, with the offset $o = 60/8063$. This translates into a time of approximately 11:58:44.9981, with hand positions

$$\left(\frac{482940}{8063}, \frac{473700}{8063}, \frac{362820}{8063}\right).$$

The hands don't overlap here, but we get another valid hand position by permuting the hands:

$$\left(\frac{473700}{8063}, \frac{362820}{8063}, \frac{482940}{8063}\right).$$

It is easy to check that this last position is indeed valid.

We get a similar solution for $o = -60/8063$.

So unless you can be certain that your second-hand is not mounted as much as $60/8063$ seconds (less than 8 thousandths of a second) off of vertical (at noon), you'd better make sure you know which hand is which!

Acknowledgment We thank the referee for pointing out Steinhaus's book [3] and improving the exposition.

References

[1] A. Gardiner, *Mathematical Puzzling*, Oxford University Press, Oxford, UK, 1987.

[2] Y. I. Perelman, *Algebra Can Be Fun* (translated from the thirteenth Russian edition), MIR Publishers, Moscow; Imported Publications, Chicago, 1979.

[3] H. Steinhaus, *One Hundred Problems in Elementary Mathematics*, Dover, New York, 1979.

[4] T. Szirtes, On the problem of the interchangeable clock hands, *J. Recreational Math.* 8 (1975/76), no. 3, 159–168.